Factoring, random maps, and polynomial maps

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Factoring is a thing that number theorists do

Example

RSA-768 =

1230186684530117755130494958384962720772853569595334
7921973224521517264005072636575187452021997864693899
5647494277406384592519255732630345373154826850791702
6122142913461670429214311602221240479274737794080665
351419597459856902143413 =
3347807169895689878604416984821269081770479498371376
8568912431388982883793878002287614711652531743087737
814467999489 · 367460436667995904282446337996279526322
7915816434308764267603228381573966651127923337341714
3396810270092798736308917

Sizes of numbers

Computers work in binary.

$$\underbrace{\overbrace{\_\_\ \_\_\ \_\_ \cdots \_\_}^{b \text{ blanks (or bits)}}}_{2^b \text{ possible numbers represented, from 0 to } 2^b-1}$$

The "size" of a number is the number of bits required to store it; ie, the size of a number n, if $2^{b-1} \leq n \leq 2^b - 1$, is

$$b(n) = \lceil \log_2 (n+1) \rceil \approx \log_2 n$$

Trial division

Suppose n is a large composite number, with smallest factor p.

Very naive factoring method (Trial Division)

Does 2 divide n?
Does 3 divide n?
Does 4 divide n?
Does 5 divide n?
Does 6 divide n?
$\vdots$

Terminates by finding p in about p steps:

$$p = 2^{\log_2 p} \approx 2^{b(p)},$$

so the runtime of Trial Division is *exponential* in $b(p)$.

Trial division

Suppose n is a large composite number, with smallest factor p.

~~Very~~ naive factoring method (Trial Division)

Does 2 divide n?
Does 3 divide n?
~~Does 4 divide n?~~
Does 5 divide n?
~~Does 6 divide n?~~
$\vdots$

Terminates by finding p in about $\frac{1}{2}p$ steps:

$$\frac{1}{2}p = \frac{1}{2}2^{\log_2 p} \approx 2^{b(p)-1},$$

so the runtime of Trial Division is (still) *exponential* in $b(p)$.

Pollard's "rho" algorithm

Pretend that I want to factor 41831.

First I need a "pseudorandom" sequence mod 41831, say

$$(x_i) = 0, 1, 2, 5, 26, 677, 40020, 16904, 39487, 14476, 23098, 5031, 3207, 36255, 11344, 14181, 19145, 7804, 38312, 1386, 38602, 10523, 6873, 10931, \ldots$$

Pollard's "rho" algorithm ($0 \to 1 \to 2 \to 5 \to 26 \to \cdots$)

What I do: compute $\gcd(41831, |x_{2i+1} - x_i|)$ for $i = 0, 1, 2, \ldots$

$\gcd(41831, |1 - 0|) = 1,$

$\gcd(41831, |5 - 1|) = 1,$

$\gcd(41831, |677 - 2|) = 1,$

$\gcd(41831, |16904 - 5|) = 1,$

$\gcd(41831, |14476 - 26|) = 1,$

$\gcd(41831, |5031 - 677|) = 1,$

$\gcd(41831, |36255 - 40020|) = 1,$

$\gcd(41831, |14181 - 16904|) = 1,$

$\gcd(41831, |7804 - 39487|) = 59.$

What's going on (Birthday Theorem)

We won when $\gcd(41831, |7804 - 39487|) = 59$; ie, when

$$x_{17} \equiv x_8 \pmod{59}.$$

Theorem (The Birthday Theorem)

Suppose you have p numbers. If you choose $\left\lceil \sqrt{\log 4} \cdot \sqrt{p} \right\rceil$ *of these numbers, with repetition, then the probability that two are the same is over* $\frac{1}{2}$*.*

So modulo 59, the odds are you'll get a repeat after

$$\left\lceil \sqrt{\log 4} \cdot \sqrt{59} \right\rceil = 11 \text{ steps.}$$

- Random number generators are not allowed
- Can't store all terms of a sequence, and wait until a repeat.

What's going on (Collision time)

The sequence (x_i) is not a random sequence. In fact, we choose a "pseudorandom" function ϕ modulo 41831, and set

$$x_0 = 0 \qquad x_i = \phi(x_{i-1})$$
$$x_0 = 0 := \phi^0(0) \qquad x_i = \phi^i(0) := \underbrace{\phi \circ \cdots \circ \phi}_{i \text{ times}}(0).$$

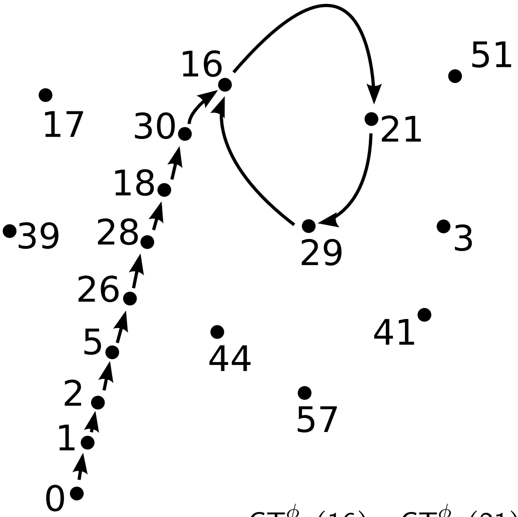
What do I mean by "collision time"?

$\mathrm{CT}^{\phi}(x_0)$ = the smallest i such that $x_i = x_j$ for some $j \le i$.

Well, we found a factor when $x_8 \equiv x_{17}$ mod 59; ie, when

$$\phi^8(0) \equiv \phi^{17}(0) \pmod{59}.$$

What's happening mod 59? $(0 \to 1 \to 2 \to 5 \to 26 \to \cdots)$



$$\mathrm{CT}^{\phi}_{59}(0) = 11$$
$$\mathrm{CT}^{\phi}_{59}(18) = 5$$
$$\mathrm{CT}^{\phi}_{59}(16) = \mathrm{CT}^{\phi}_{59}(21) = \mathrm{CT}^{\phi}_{59}(29) = 3$$

Collision time (continued)

Theorem (Collision time)

The average collision time of random functions on size p set is $\sqrt{p}$. More precisely, there are p^p functions on $\{0, \ldots, p-1\}$ and

$$\frac{1}{p^p} \cdot \sum_{f:\{0,\ldots,p-1\}\to\{0,\ldots,p-1\}} \left(\frac{1}{p} \cdot \sum_{a\in\{0,\ldots,p-1\}} \mathrm{CT}_p^f(a) \right) \sim_p \sqrt{p}.$$

And now something amazing happens

Let $\phi(x) = x^2 + 1$.

Floyd's cycle-finding algorithm

Algorithm is deterministic.

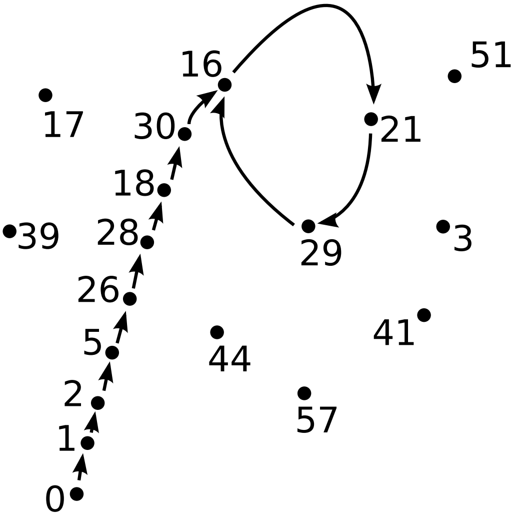
How do we find collisions? (Recall, we can't just "remember" everything that has happened.)

We use Floyd's cycle-finding algorithm:

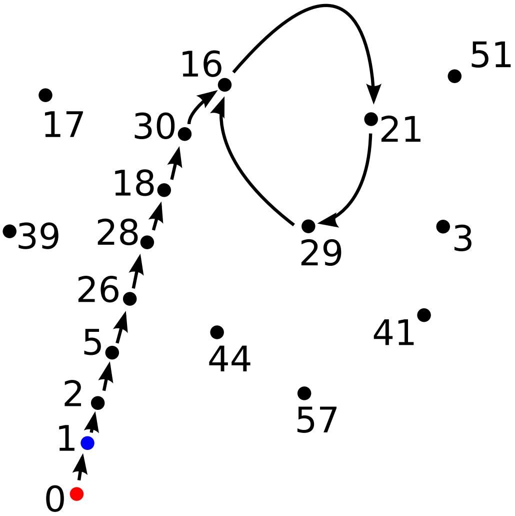
Algorithm (Floyd's cycle finding algorithm)

Compute $|x_{2i+1} - x_i|$ *for* $i = 0, 1, 2, \ldots$

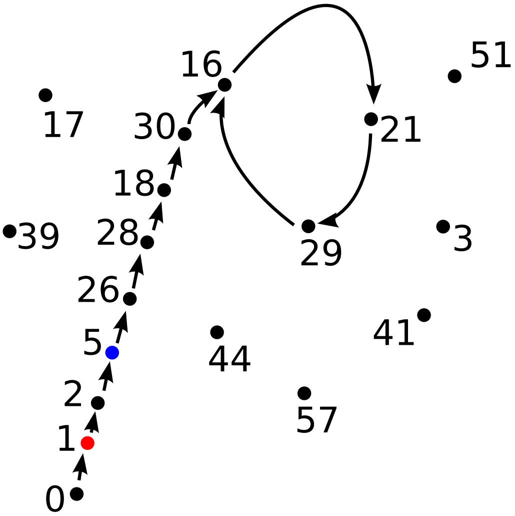
What's happening mod 59?



What's happening mod 59?



What's happening mod 59?



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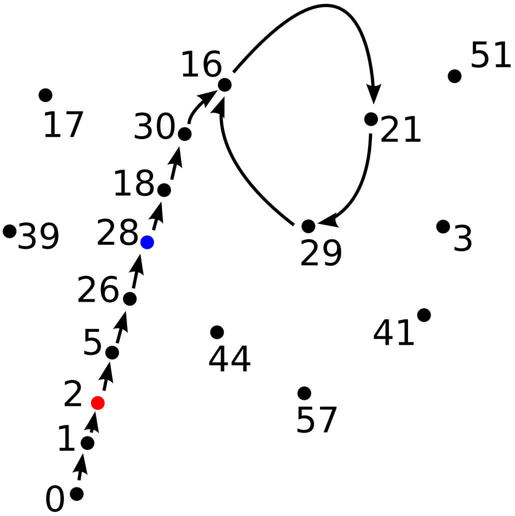

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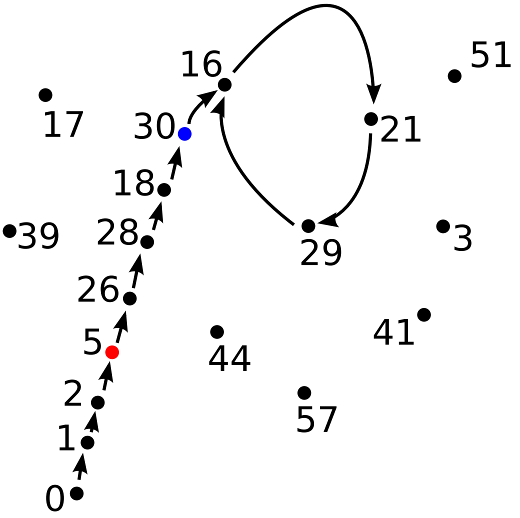

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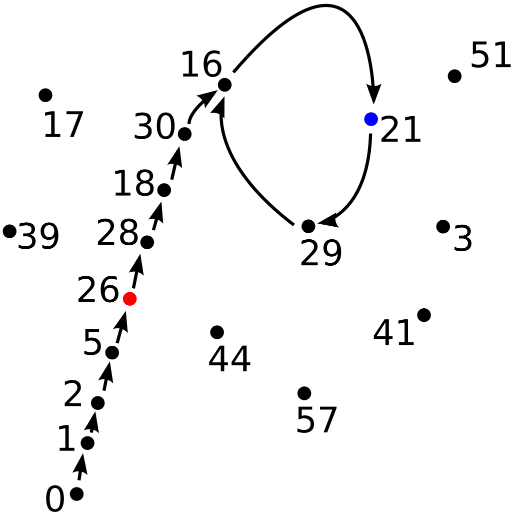
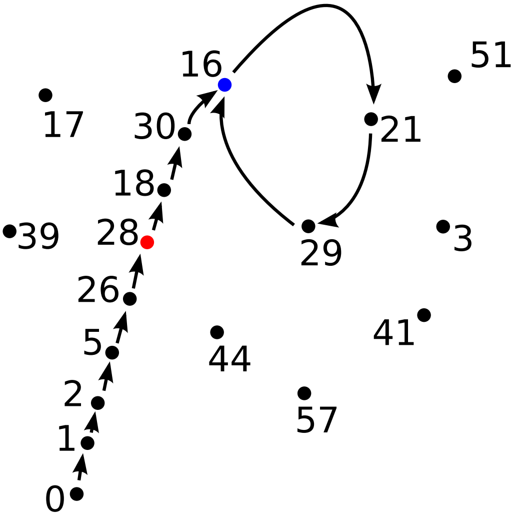


What's happening mod 59?



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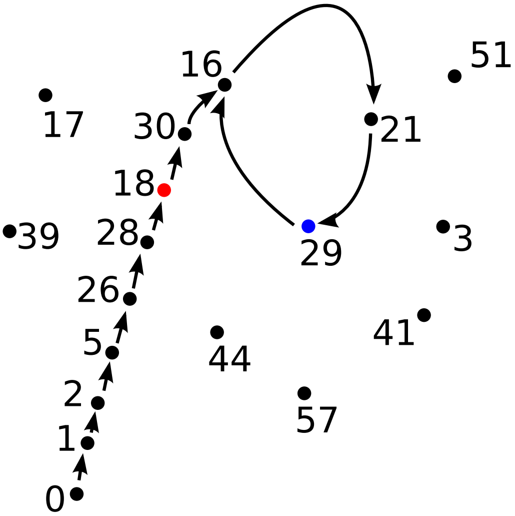

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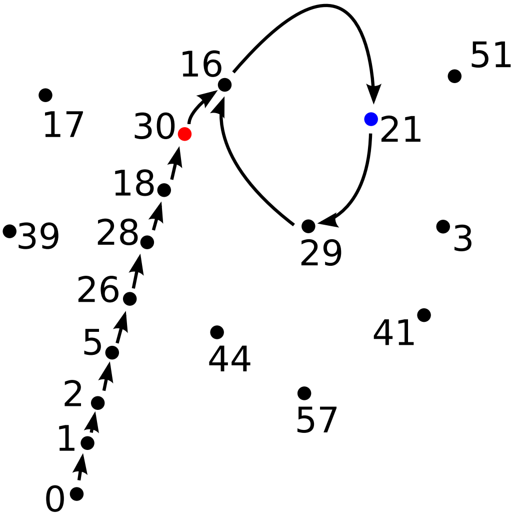

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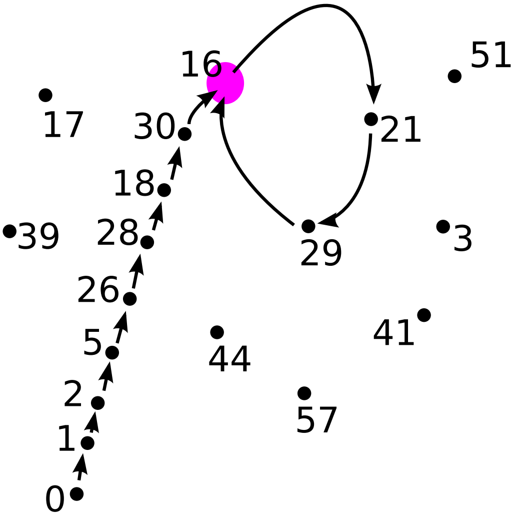

Failure (and conjectural success)

Pollard's "rho" algorithm can fail, just try to factor 4:

$$0 \to 1 \to 2 \to 1 \to 2 \to 1 \to 2 \to 1 \to \cdots$$

Success Conjecture (Silverman)

Let $\phi(x)$ be the polynomial $\phi(x) = x^2 + 1$, and let $\epsilon > 0$. Then.

$$\lim_{X\to\infty} \frac{|\{p \leq X \text{ with ``correct'' CT}\}|}{|\{p \leq X\}|} = \lim_{X\to\infty} \frac{\left|\left\{p \mid \mathrm{CT}_p^{\phi}(0) \leq p^{\frac{1}{2}+\epsilon}\right\}\right|}{|\{p \leq X\}|} = 1$$

Runtime

Success Conjecture

So when the "rho" algorithm works, its runtime is about

$$\sqrt{p} = 2^{\log_2 \sqrt{p}} = 2^{\frac{1}{2}\log_2 p} = 2^{\frac{1}{2}b(p)} = \sqrt{2}^{b(p)}.$$

Better!

Part II: The Arizona Winter School 2010

- Week-long number theory workshop every March.
- It's warm there.
- 2010 was the "dynamics" year.
- Silverman was my project leader.
- Silverman said, "Remember how every polynomial gives you a graph $\bmod p$ for every prime p?"
- "Can you say anything about the average number of components of these graphs?"

Comparing polynomials to random functional graphs

Facts about random functional graphs graphs

- *The average number of components of a random functional graph on p numbers is* $\sim_p \frac{1}{2}\log p$.
- *The average number of periodic points of a random functional graph on p numbers is* $\sim_p \sqrt{p}$.
- *The average collision time of a random functional graph on p numbers is* $\sim_p \sqrt{p}$.

There are two ways to show polynomial functions "act randomly":

- Fix a polynomial ϕ and consider ϕ mod p for all primes p.
- Fix a prime p, and consider all polynomials ϕ (of a fixed, hopefully small, degree d) over $\mathbb{F}_p$.

The "other way" to average polynomial maps

Goal

Fix a prime p and a degree d. Compute

$$\frac{1}{\left|\{\phi \in \mathbb{F}_p[x] \mid \deg(\phi) = d\}\right|} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi) = d}} \left|\{\text{components of } \Gamma_\phi\}\right|.$$

In particular, is the above average anywhere close to $\frac{1}{2} \log p$?

Randomness result

Theorem (Flynn and G.)

$$\frac{1}{\left|\left\{\phi \in \mathbb{F}_p[x] \mid \deg(\phi) = d\right\}\right|} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi)=d}} \left|\left\{\textit{components of } \Gamma_\phi\right\}\right|$$

$$> \log\left(\min\left\{d, \sqrt{p}\right\}\right) - \frac{1}{4}.$$

In particular, if $d \geq \sqrt{p}$, then

$$\frac{1}{\left|\left\{\phi \in \mathbb{F}_p[x] \mid \deg(\phi) = d\right\}\right|} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi)=d}} \left|\left\{\textit{components of } \Gamma_\phi\right\}\right|$$

$$> \frac{1}{2}\log(p) - \frac{1}{4}.$$

A sample computation

Any polynomial looks like of degree d or less has the form

$$\phi(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

so there are

- p^{d+1} polynomials of degree d or less,
- p^d polynomials of degree $d-1$ or less, and
- $p^{d+1} - p^d$ polynomials of degree d.

We try to compute

$$\frac{1}{p^{d+1} - p^d} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi) = d}} \left| \left\{ \text{components of } \Gamma_\phi \right\} \right|.$$

Lagrange interpolation

Lagrange interpolation

Suppose you have k data points in $\mathbb{F}_p$; ie, a set $S = \{(a_1, b_1), \ldots, (a_k, b_k)\}$.

Then there exists an interpolating polynomial ϕ_S with $\deg(\phi_S) < k$;
ie, such that $\phi_S(a_i) = b_i$ for all $i \in \{1, \ldots, k\}$.

This is the analog of the "theorem" that two points determine a line; ie, given (x_1, y_1) and (x_2, y_2), that line is

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

so $\phi_S(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2) + y_2$.

Examples of Lagrange interpolation

- Suppose $S = \{(2,2),(4,6),(5,2)\} \subseteq \mathbb{F}_7 \times \mathbb{F}_7$.
  Then $\phi_S(x) = 5x^2 + 3$, and

$$\begin{aligned} \phi_S(2) &= 2 \\ \phi_S(4) &= 6 \\ \phi_S(5) &= 2. \end{aligned}$$

- Suppose $S = \{(1,3),(2,0),(5,3),(6,1)\} \subseteq \mathbb{F}_7 \times \mathbb{F}_7$.
  Then $\phi_S(x) = x^2 + x + 1$, and

$$\begin{aligned} \phi_S(1) &= 3 \\ \phi_S(2) &= 0 \\ \phi_S(5) &= 3 \\ \phi_S(6) &= 1. \end{aligned}$$

Idea of proof

Instead of counting "components per polynomial", count "polynomials per component" instead.

That is,

$$\sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi)=d}} \left|\left\{\text{components of } \Gamma_\phi\right\}\right|$$

$$= \sum_{\substack{\text{possible} \\ \text{components } C}} \left|\left\{\text{polynomials } \phi \text{ interpolating } C\right\}\right|.$$

Observation

Cycles are in 1-1 correspondence with graph components, so let's count cycles instead.

Making the counting problem easier

- Count the number of possible cycles of length k.
- Count the number of polynomials interpolating each k-cycle.
- Then

$$\begin{aligned}
&\sum_{\substack{\phi\in\mathbb{F}_p[x]\\ \deg(\phi)=d}} \left|\left\{\text{components of } \Gamma_\phi\right\}\right| \\
&= \sum_{\substack{\text{possible}\\ \text{components } C}} \left|\left\{\text{polynomials } \phi \text{ interpolating } C\right\}\right| \\
&= \sum_{\substack{\text{possible}\\ \text{cycles } C}} \left|\left\{\text{polynomials } \phi \text{ interpolating } C\right\}\right| \\
&= \sum_{k=1}^{p} \left(\begin{matrix}\text{number of cycles}\\ \text{of length } k\end{matrix}\right)\left(\begin{matrix}\text{number of polynomials}\\ \text{interpolating a } k\text{-cycle}\end{matrix}\right).
\end{aligned}$$

The final step of the proof (and recap)

$$\frac{1}{\left|\left\{\phi \in \mathbb{F}_p[x] \mid \deg(\phi) = d\right\}\right|} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi)=d}} \left|\left\{\text{components of } \Gamma_\phi\right\}\right|$$

$$= \frac{1}{p^{d+1} - p^d} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi)=d}} \left|\left\{\text{components of } \Gamma_\phi\right\}\right|$$

$$= \frac{1}{p^{d+1} - p^d} \cdot \sum_{\substack{\text{possible} \\ \text{components } C}} \left|\left\{\text{polynomials } \phi \text{ interpolating } C\right\}\right|$$

$$= \frac{1}{p^{d+1} - p^d} \cdot \sum_{\substack{\text{possible} \\ \text{cycles } C}} \left|\left\{\text{polynomials } \phi \text{ interpolating } C\right\}\right|$$

$$= \frac{1}{p^{d+1} - p^d} \cdot \sum_{k=1}^{p} \binom{\text{number of cycles}}{\text{of length } k} \binom{\text{number of polynomials}}{\text{interpolating a } k\text{-cycle}}$$

$$> \frac{1}{p^{d+1} - p^d} \cdot \sum_{k=1}^{\min\{d, \sqrt{p}\}} \left(\frac{p^k}{k}\right)\left(p^{d-k+1} - p^{d-k}\right) = \sum_{k=1}^{\min\{d, \sqrt{p}\}} \frac{1}{k}. \quad \square$$

The theorem

Theorem (Flynn and G.)

$$\frac{1}{\left|\{\phi \in \mathbb{F}_p[x] \mid \deg(\phi) = d\}\right|} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi) = d}} \left|\{\textit{components of } \Gamma_\phi\}\right| > \log\left(\min\{d, \sqrt{p}\}\right) - \frac{1}{4}.$$

In particular, if $d \geq \sqrt{p}$, then

$$\frac{1}{\left|\{\phi \in \mathbb{F}_p[x] \mid \deg(\phi) = d\}\right|} \cdot \sum_{\substack{\phi \in \mathbb{F}_p[x] \\ \deg(\phi) = d}} \left|\{\textit{components of } \Gamma_\phi\}\right| > \frac{1}{2}\log(p) - \frac{1}{4}.$$

Further work

We only count k-cycles for $k \leq d$. The long cycles are mysterious.

What about when $d + 1 < k < \sqrt{p}$?

- *Well, we bound the average below, by* $\sum_{k=1}^{d} \frac{1}{k} \approx \log d$.
- *But we want a better bound!*
- *How many degree d polynomials interpolate a k-cycle when $k > d$?*
- *The odds that three points are on a line is* $\frac{1}{p}$.
- *The odds that four points are on a line is about* $\frac{1}{p^2}$.
- *The odds that k points are on a degree d polynomial... is*
$$p^{d+1-k}?$$
- *Do these probabilities carry over to cycles?*
- *Do these probabilities give the right answer?*

More further work

What about when $k > \max\{\sqrt{p}, d+1\}$?

- *We only bound the average from below, what about from above?*
- *When $k > \sqrt{p}$, the number of cycles is "about" zero.*
- *Is it "zero enough" to give an upper bound, assuming the guess on the previous slide?*
- *We could carry out computations to test the hypothesis that degree 2 polynomials act like random functional graphs.*

Thank you slide

Thank you!