Dependence Patterns across Financial Markets: a Mixed Copula Approach*

Ling Hu†

June 2004

Abstract

We study the modeling and estimation of associations across international financial markets, with a focus on the *structure* of dependence. A mixed copula model is constructed so that it can capture various patterns of dependence structures. The marginal distribution of asset returns in each market is estimated nonparametrically and a quasi-ML method is used to estimate the mixed copula. The methodology is applied to estimate the dependence across several international stock markets. The empirical findings are shown to have some implications that seem important for a wide range of multivariate studies in Economics and Finance.

Keywords: Asset return, copulas, dependence modeling, international markets, mixture models.

JEL Classifications: C51, G15

*An early version of the paper is under the title "Dependence Patterns across Financial Markets: Methods and Evidence".

†Corresponding author: Department of Economics, The Ohio State University, 1945 N. High Street, RM 410, Columbus, OH 43210. Phone: (614) 292-4198, E-mail: linghu@econ.ohio-state.edu.

1 Introduction

Dependence across financial markets has been widely studied in the past decades. Three general alternatives are available in multivariate analysis for studying dependence[1]. One approach is to use a joint distribution, with the most commonly used joint distribution in theoretical and empirical finance being the multivariate normal distribution. Under Gaussian assumptions[2], inference can then be conveniently based on mean-variance analysis. However, there is increasing evidence indicating that Gaussian assumptions are inappropriate in the real world, both in domestic markets (e.g. Ang and Chen, 2002) and in international markets (e.g. Longin and Solnik, 2001, Ang and Bekaert, 2002).

The key statistics in a joint Gaussian distribution is the correlation coefficient. Typically, a low correlation coefficient between two markets implies a good opportunity for an investor to diversify her investment risk. For instance, suppose that the annual returns in a domestic market and in a foreign market have a linear correlation coefficient of 0.2. Under the Gaussian assumption, the probability that the returns in both markets are in their lowest 5th percentiles is less than 0.005, or about 5 times in a millennium. Hence based on the Gaussian assumption, an investor can significantly reduce her risk by balancing her portfolio with investments in the foreign market. However, it has been widely observed that market crashes and financial crisises often happen in different countries during about the same time period, even when the correlation between these markets are fairly low. In the example, instead of observing those events 5 times in a millennium, we may observe them more than once in a century. These observations raise a question: in multivariate studies, it is not only the degree of dependence that matters, but also the *structure* of dependence matters. Pairs of markets with the same correlation coefficient could have very different dependence structures and these different structures could increases or decrease the diversification benefit compared to the Gaussian assumption. Embrechts *et al.* (2002) illustrates the pitfalls and limitations of using linear correlation coefficients to study dependence.

A second approach that has been used in empirical study is to compute conditional correlations. It has been found that correlations computed with different conditions could differ dramatically. Ang *et al.* (2002) study the correlations between a portfolio and the market conditional on downside movements, and they find that downside correlations help to explain expected returns of a portfolio. It has also been found that correlations conditional on large movements are higher than that conditional on small movements. This phenomenon has also been characterized as "correlation breakdown", and it is widely documented in the literature of contagion (say, Forbes and Rigobon, (2002)). But Boyer *et al.* (1999) proposed that in such situations correlations can reveal little about the underlying nature of dependence. This is because even a stationary Gaussian process predicts stronger dependence in volatile periods and weaker dependence in tranquil periods. Therefore, although conditional correlations provide more information about the dependence than unconditional correlations, the results are sometimes misleading and need to be interpreted with caution.

A third alternative approach in multivariate analysis is to use a *copula* model for directly modeling dependence, and this is the approach we adopt in this paper. Compared to the joint distribution approach or correlation-based approach, a copula model is a more convenient tool in studying the dependence structure. In statistics, a copula is a function that connects the marginal distribu-

[1] There are other approaches which are not discussed here. For example, Karolyi and Stulz (1996) and Bae *et al.* (2003) use regressions to model the co-movements across markets.

[2] In this paper, a Gaussian assumption is the assumption of joint Gaussian distribution. This needs to be distinguished from a Gaussian copula, which not necessarily generates a Gaussian distribution unless the marginals are also Gaussian. On the other hand, the copula corresponding to a Gaussian distribution is a Gaussian copula. This will be clear later in the paper.

tions to restore the joint distribution and various copula functions represent various dependence structures between variables. In a copula model, the primary task is to choose an appropriate copula function and a corresponding estimation procedure. Marginal distributions are treated as nuisance functions. This reorientation has desirable advantages in empirical finance where one of the primary goals is to investigate dependence in order to better understand issues like portfolio allocation and where the marginal distributions of asset returns in individual markets may be very complicated and may not easily fit within existing parametric models.

Despite their long history in statistics, copulas have been applied to financial markets only very recently. But research in this area grows very fast. Bouye *et al.* (2000), Embrechts *et al.* (2002) and Embrechts *et al.* (2003) provide some general guide to applying copulas in finance, especially in risk management. Li (2000) use a copula function to study the default correlations in credit risk models. Among the empirical works, Longin and Solnik (2001) use a Gumbel copula to estimate the extreme correlations across international equity markets; Mashal and Zeevi (2002) estimate the degree of freedom in a t copula and use it to test the Gaussian assumption in financial markets.

In the present paper, we will estimate the dependence structure in several major stock markets using a mixed copula approach. We aim to find a simple yet flexible model to summarize the dependence structure. The mixture is composed of a Gaussian copula, a Gumbel copula and a Gumbel survival copula. A Gaussian copula has zero tail dependence, i.e., the probability that both variables are in their extremes is asymptotically zero unless their linear correlation coefficient is unit. We keep a Gaussian copula in the mixture for the purpose of connecting our approach with traditional approaches and theoretical treatments based on Gaussian assumption and it serves as a benchmark for comparison. Furthermore, although we will not explicitly test the hypothesis of a Gaussian dependence structure, the estimated weight on the Gaussian component is certainly informative about its role in modeling the overall dependence structure.

To take into account of possible tail dependence, we add a Gumbel copula and its survival copula in the mixture. A Gumbel copula has a positive right tail dependence, which means the probability that both variables are in their right tails is positive. A Gumbel survival copula is its mirror image and has positive left tail dependence: the probability that both variables are in their left tails is positive. Intuitively, this mixture model improves a Gaussian dependence structure by allowing possible asymmetric tail dependence.

This new method also facilitates the separation of the concepts of degree of dependence and structure of dependence, and these concepts are embodied in two different groups of parameters – association parameters and weight parameters. The association parameters are the parameters in each copula functions that control the degree of dependence, while the weight parameters reflects the shape of the dependence. This separation is useful in the context of testing for stability of the dependence, when it would be possible to have a switch in the dependence structures over time while the degree of dependence is largely unchanged. The opposite is also possible.

The dataset in the paper contains four stock market indices from different regions over about 30 years, in monthly frequency. We did not find any right tail dependence among all the pairs, which means these markets are not particularly likely to boom together. We find that a Gaussian copula is appropriate only in one of the six pairs, and we find significant left tail dependence in all the remaining five pairs. The findings suggest that tail dependence does exist and it is asymmetric – the markets in different regions are much more likely to crash together than to boom together. In other words, good story is individual, while bad story is worldwide.

Based on the empirical findings in the paper, we suggest some implications in portfolio choice, risk management and asset pricing. In the first place, use of multivariate normality and correlation coefficients to measure dependence may significantly underestimate market risk. We compare value at risk, a commonly used measure of risk, computed using the Gaussian assumption and using the estimated mixed copula. In almost all cases we find that the VaR computed under the Gaussian assumption underestimates the risk and the VaR computed using the mixed copula is much more realistic. Next, in hedging risk and in asset diversification dependence structures as well as degree should enter into market valuation. Finally, other things being equal, assets should be priced differently if they have different dependence structures with the aggregate market.

The paper is organized as follows. Section 2 reviews some basic concepts about copula, and introduces the mixture model. Section 3 describes the estimation and inference procedure. Section 4 presents the empirical results. Section 5 concludes.

2 The Model

2.1 The Copula Concepts and Alternative Measures of Dependence

In the statistics literature, the idea of a copula arose as early as the 19th century in the context of discussions of non-normality in multivariate cases. Modern theories about copulas can be dated to about forty years ago when Sklar (1959) defines and provides some fundamental properties of a copula:

Theorem 1 *Suppose that H is a distribution function on R^k with one dimensional distribution $F_1, \ldots, F_k$, then there is a copula C so that*

$$\begin{aligned} H(x_1, \ldots, x_k) &= P[F_1(X_1) \leq F_1(x_1), \ldots, F_k(X_k) \leq F_k(x_k)] \\ &= C(F_1(x_1), \ldots, F_k(x_k)). \end{aligned} \quad (1)$$

If H is continuous, then the C in (1) is unique and is given by

$$C(u_1, \ldots, u_k) = H(F_1^{-1}(u_1), \ldots, F_k^{-1}(u_k))$$

for $u = (u_1, \ldots, u_k) \in R^k$ where $F_i^{-1}(u) = \inf\{x : F_i(x) \geq u\}$, $i = 1, \ldots, k$. Conversely, if C is a copula on $[0,1]^k$ and $F_1, \ldots, F_k$ are distribution functions on R, then the distribution function defined in (1) is a distribution function on R^k with one-dimensional marginal distribution $F_1, \ldots, F_k$.

It is clear that the copula is a map from $[0,1]^k$ to $[0,1]$. The copula is invariant under increasing and continuous transformations[3]. This property is very useful as transformations are commonly used in economics. For example, no matter whether we are working with price series or with log price series, we have the same copula. To avoid confusions, we will use symbols x and y to denote the observations of random variables X and Y; and we will use symbols u and v to denote their marginal CDFs. So x and y could be any real numbers but u and v must be in $[0,1]$. A simple example of a copula is an independent copula, which is defined as

$$C(u, v) = u \cdot v.$$

Suppose one asks what is the probability that both returns in market A and in market B are in their lowest 20th percentiles? If these two markets are independent, using the independent copula, we have

$$C(u, v) = C(0.2, 0.2) = 0.2 \cdot 0.2 = 0.04,$$

[3]For detailed explanations of the notations introduced in this section, please refer to Joe (1997), Nelson (1999) or Bouye *et al.* (2000).

which is simple and intuitive. More complicated copula functions usually contain one or more parameters, which are also called association parameters. If only one parameter presents in a copula function, this parameter usually reflects the strength of the dependence.

The density (PDF) of a copula $C(u, v)$ is given by

$$c(u, v) = \frac{\partial C(u, v)}{\partial u \partial v}. \tag{2}$$

For the example of independent copula, $c(u, v) = 1$, implying that the dependence is flat over $[0, 1]^2$. Another notation we will use in this paper is a survival copula of $C(u, v)$, denoted by $C_{\mathrm{S}}(u, v)$:

$$C_{\mathrm{S}}(u, v) = u + v - 1 + C(1 - u, 1 - v). \tag{3}$$

The density of the survival copula and the density of the original copula are related by

$$c_{\mathrm{S}}(u, v) = c(1 - u, 1 - v).$$

Hence they are mirror images about $(u, v) = (1/2, 1/2)$.

Compared to working with the joint distribution $H(X_1, \ldots, X_n)$ directly, working with the copula model has several advantages. First, it is more flexible in applications. In many cases, it maybe hard to specify a joint distribution directly when the scatter of the data does not fit within any existing families. Using the copula approach, we can first estimate the marginal distributions, and then estimate the copula. This two-step approach gives the investigator more options in model specifications. Second, in a copula model approach, we obtain a dependence function explicitly, which enables us to provide a more delicate description of dependence. Since the description of dependence can be critical in financial analysis, this capability is important. For example, the market price of an asset is largely determined by its co-movement with the aggregate market.

Besides linear correlations, there are several other popular measures of dependence, among which Spearman's ρ_s and Kendall's τ are commonly studied together with the copula model. Both of them are rank correlations. As implied by their name, rank correlations reflect relations between the rankings, rather than the actual value of the observations. In this paper, we will mostly use Kendall's τ, which is defined as

$$\tau(X, Y) = \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) < 0],$$

i.e., $\tau(X, Y)$ is the probability of concordance minus the probability of discordance of variables (X, Y). Similar as a linear correlation coefficient, $\tau \in [-1, 1]$ and a positive τ implies positive dependence with the higher the value, the stronger the dependence. The relation between Kendall's τ and a copula is given in the following moment condition:

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1. \tag{4}$$

Hence for a copula with single parameter, the parameter is determined if τ is fixed.

Another important notation is tail dependence, which measures the probability that both variables are in their extremes. The ability to capture tail dependence is an important property of a copula. A copula $C(u, v)$ is said to have left (lower) tail dependence if

$$\lim_{u \to 0} C(u, u)/u = \lambda_l > 0. \tag{5}$$

To define the right (upper) tail dependence, we need the notation of a joint survival function[4],

$$\bar{C}(u, v) = 1 - u - v + C(u, v). \tag{6}$$

A copula $C(u, v)$ measures the probability that its two arguments are below value u and v respectively, while its joint survival function measures the probability that these two arguments are above value u and v respectively. Now, a copula $C(u, v)$ is said to have right (upper) tail dependence if

$$\lim_{u \to 1} \bar{C}(u, u)/(1 - u) = \lambda_r > 0. \tag{7}$$

2.2 The Mixture Model

We will consider three copulas. First, let $\rho \in [-1, 1]$ and let $\mathbf{\Phi}_\rho$ be the standardized bivariate normal distribution with correlation ρ, the bivariate Gaussian copula is defined as

$$C_{\text{Gaussian}}(u, v; \rho) = \mathbf{\Phi}_\rho \left(\Phi^{-1}(u), \Phi^{-1}(v) \right), \tag{8}$$

where Φ is the univariate normal distribution. This notation can be easily extended to multivariate cases, with ρ replaced by a correlation matrix. The higher the association parameter ρ, the stronger the dependence. In particular, for all elliptical distributions, we have the following relationship:

$$\rho = \sin\left(\frac{\pi}{2} \tau \right). \tag{9}$$

A Gaussian copula represents the dependence structure of a joint Gaussian distribution, but a Gaussian copula does not necessarily imply a joint Gaussian distribution unless the marginals are also Gaussian.

The first column in Figure 1 plots the density of a Gaussian copula[5]. The upper graph plots the contours of the density of the copula, and the lower graph plots the cross section of the density on the diagonal $u = v$, i.e., when the observations in two series take the same percentiles (e.g., both are below 5th percentile, 10th percentile, etc). Clearly a Gaussian dependence structure is symmetric and in the example of market returns, a Gaussian copula implies that two markets are equally likely to boom together and to crash together.

Note that the shape in the lower graph can be summarized using letter *U*. We have mentioned a phenomenon "correlation breakdown", which means that correlations conditional on large movements are higher than that conditional on small movements. And Boyer *et al.* (1999) proposed that in such situations correlations can reveal little about the underlying nature of dependence. This is because that a Gaussian assumption itself implies higher correlations conditional on large movements, as shown in this *U* shape. Hence, the fact that higher dependence in volatile periods is not necessarily "abnormal". We need a benchmark to make the judgments and a natural benchmark in many situations is Gaussian. If the dependence in volatile periods is not only higher than that is computed in tranquil periods, but also significantly higher than that is implied from Gaussian, this is an evidence for contagion and against a Gaussian assumption.

---

[4]The survival function needs to be distinguished from the survival copula. For a copula $C(u, v)$, the relation between its survival copula $C_{\text{S}}(u, v)$ and its survival function $\bar{C}(u, v)$ is

$$C_{\text{S}}(u, v) = \bar{C}(1 - u, 1 - v)$$

which can be seen from the numerical example in the next section.

[5]We set Kendall's τ to be 0.3 in all the plots in this Figure.

The second copula we consider is a Gumbel copula:

$$C_{\text{Gumbel}}(u, v; \alpha) = \exp\left\{-\left[(-\log(u))^{1/\alpha} + (-\log(v))^{1/\alpha}\right]^{\alpha}\right\}, \quad \alpha \in (0, 1]. \tag{10}$$

Here the association parameter is α, with the larger the α, the weaker the dependence. This can be seen more clearly from the relation

$$\alpha = 1 - \tau, \tag{11}$$

which can be computed from equation (4). The second column in Figure 1 plots the contours of the density (upper graph) and the density on the diagonal $u = v$ (lower graph). It is clear that the Gumbel copula is asymmetric about $(1/2, 1/2)$. It puts more density on the right tails. The shape of the density plot on the diagonal is similar to letter J. Using the example of market returns, a J dependence shape implies that two markets are more likely to boom together rather than to crash together.

The third copula function we will use in the mixture is a Gumbel survival copula (GS for short), which can be written as

$$C_{\text{GS}}(u, v; \beta) = u + v - 1 + \exp\left\{-\left[(-\log(1-u))^{1/\beta} + (-\log(1-v))^{1/\beta}\right]^{\beta}\right\}, \quad \beta \in (0, 1]. \tag{12}$$

We already know that the density of a Gumbel copula is asymmetric, hence by including its mirror image (about $(1/2, 1/2)$), the mixture is able to capture dependence structures with possible symmetric and asymmetric tail dependence. Since the density of the GS and the Gumbel are mirror of each other, for the same Kendall's τ, $\alpha = \beta = 1 - \tau$.

The third column in Figure 1 plots the contours of the density (upper graph) and the density on the diagonal $u = v$ (lower graph) of a GS copula. It is clear that it puts more density on the left tails and the shape of the density plot on the diagonal is similar to letter L. Using the example of market returns, an L dependence implies that two markets are more likely to crash together rather than to boom together.

We can compute tail dependence coefficients λ_l and λ_r for these three copulas using formula (5) and (7). It can be shown that the Gaussian has zero tail dependence: $\lambda_l = \lambda_r = 0$, and the Gumbel has positive right tail dependence: $\lambda_l = 0$ and $\lambda_r = 2 - 2^{\alpha}$, and finally, the GS has positive left tail dependence: $\lambda_r = 0$ and $\lambda_l = 2 - 2^{\beta}$.

Let's consider a numerical example to see the difference of tail dependence between these three structures. Suppose that Kendall's τ between the annual returns in two markets is 0.5. From (9) and (11), we can compute that $\rho = 0.71$, $\alpha = 0.5$ and we also have $\beta = \alpha = 0.5$. If we want to know the probability that both returns are in their lowest 5th percentiles and the probability that both returns are in their highest 5th percentiles, we can compute

$$\begin{aligned}
&\text{Gaussian} \quad C(0.05, 0.05; 0.71) = 0.0181, \quad \bar{C}(0.95, 0.95; 0.71) = 0.0181;\\
&\text{Gumbel} \quad C(0.05, 0.05; 0.5) = 0.0145, \quad \bar{C}(0.95, 0.95; 0.5) = 0.0300;\\
&\text{GS} \quad C(0.05, 0.05; 0.5) = 0.0300, \quad \bar{C}(0.95, 0.95; 0.5) = 0.0145,
\end{aligned}$$

where $\bar{C}(u, v)$ is defined as in (6). So, according a Gaussian dependence structure, the probability that both markets are in their lowest 5th percentiles is 0.018 or less then twice in a century; while according a Gumbel survival copula, the probability of such an event is 0.03, or about three times in a century.

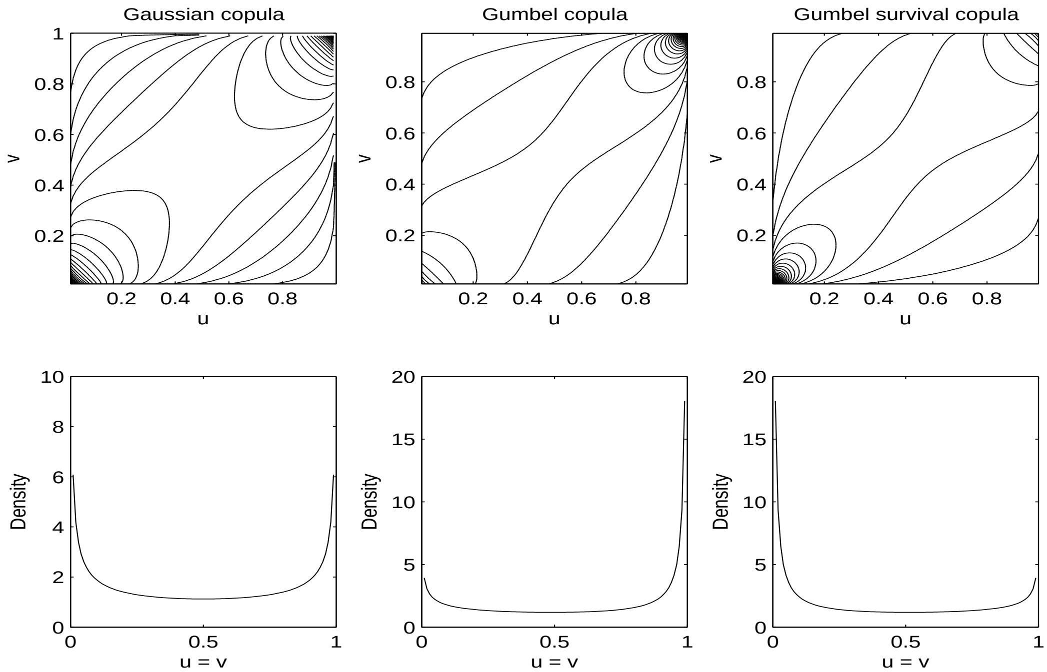

Figure 1: Contour plots of the density of three copulas and their cross sections on the diagonal $u = v$.

With these three copulas, we are ready to define a mixture model. Take $w_1, w_2 \in [0, 1]$ with $w_1 + w_2 \leq 1$ and define a mixed copula as

$$C_{\mathrm{mix}}(u, v; \theta, \mathbf{w}) = w_1 C_{\mathrm{Gaussian}}(u, v; \rho) + w_2 C_{\mathrm{Gumbel}}(u, v; \alpha) + (1 - w_1 - w_2) C_{\mathrm{GS}}(u, v; \beta), \tag{13}$$

where $\theta = (\rho, \alpha, \beta)$ are *association parameters* in the mixture which reflect the degree of dependence, and $\mathbf{w} = (w_1, w_2)$ are weight or *shape parameters* which reflect the dependence structures. In Figure 2, we plot some mixtures as examples. For the left one, $w_1 = 1/2, w_2 = 0$; for the middle one, $w_1 = w_2 = 1/3$; for the right one, $w_1 = 0, w_2 = 1/2$. Compared to individual copula functions, the mixture is much more flexible in modeling various dependence structures.

3 The Estimation Methods

There are usually two approaches to estimate a copula model. One approach is to estimate the copula and marginal distributions jointly. In this approach, the estimation of marginals will affect the estimation of the copula, and vice versa. The computation will also be of concern if both copula and marginals take some complicated form. The second approach is a two-stage estimation. First, we estimate the marginals, either in a parametric way or nonparametric way, assuming independence. Then plug the estimated marginal functions into the copula and use Maximum Likelihood (ML) or General Method of Moment (GMM) to estimate the parameters in the copula. The whole approach can be parametric (if the marginal is estimated parametrically) or semiparametric (if the marginal is estimated nonparametrically).

In this paper, we estimate the mixture model in a two-stage semiparametric way. One advantage of this approach is that we do not specify the marginals, hence the approach is robust and free of

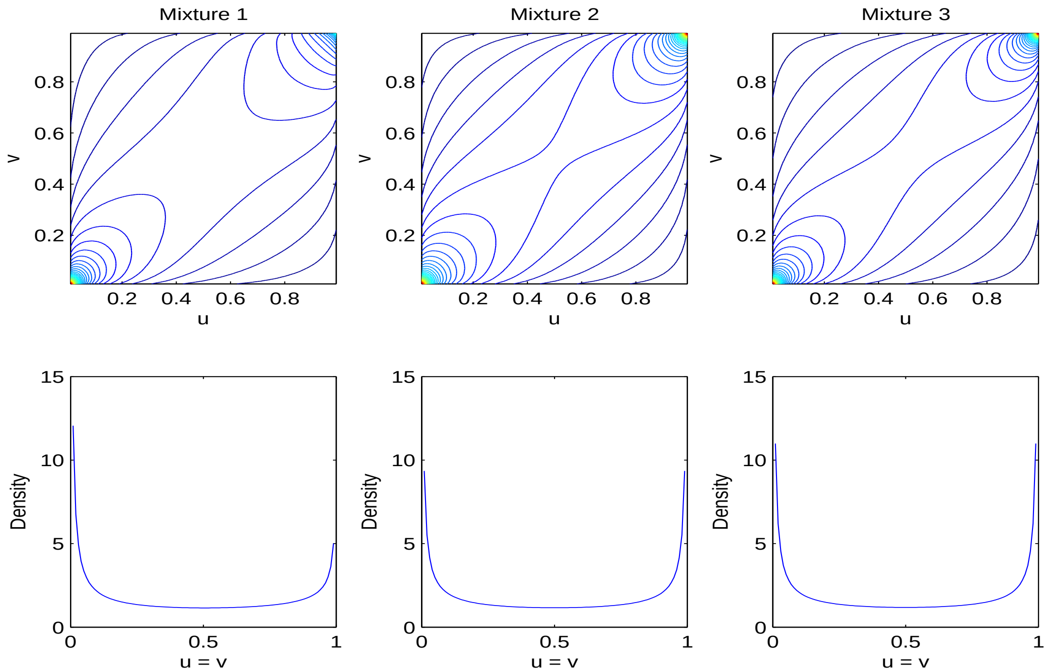

Figure 2: Contour plots of the density of three mixed copulas and their cross sections on the diagonal $u = v$.

specification errors from the marginals. Let a sample $(Z_1, Z_2, \ldots, Z_n)$ be generated independently from a univariate distribution $F_Z(z)$, the empirical CDF of Z can be computed using

$$\hat{F}_Z(z) = \frac{1}{n}\sum_{t=1}^{n} \mathbf{1}\{Z_t \leq z\}.$$

Let $\hat{F}_X(x)$ and $\hat{F}_Y(y)$ denote the empirical CDFs of X and Y, the joint distribution can be written in the form

$$H(x, y; \theta) = C(\hat{F}_X(x), \hat{F}_Y(y); \theta)$$

where θ denotes the parameters in the copula. From this expression, the distribution now becomes a function of $\hat{F}_X(x)$ and $\hat{F}_Y(y)$.

To estimate the parameter, we use Maximum Likelihood estimation. The inference procedure we use in this paper is based on the asymptotic distributions derived in Genest *et al.* (1995), which shows that under some regularity conditions, the ML estimators in this semiparametric setup are consistent and asymptotically normal.

One problem with the empirical estimation is that the financial return data is usually not i.i.d. but conditional heteroskedastic, while the asymptotic results in Genest *et al.* (1995) is for i.i.d. observations. Hence a GARCH filter is applied before empirical distributions are computed[6]. We conduct a Monte Carlo experiment to study the effects of the pre-filtering on the estimation. We

[6]Engle (2002) proposes a multivariate GARCH model with dynamic conditional correlations. A two-stage procedure is proposed in the ML estimation. First, a GARCH filter is applied to each individual sequence; second, the parameters involved in the dynamics of the correlation are estimated using the residuals from the first step. This is similar to the pre-filtering we adopted in this paper.

take the Gumbel copula as an example, and set the sample size to be 500, and the number of iterations to be 1000. The true value of α is 0.5. In Figure 3, the solid line draws the density of ML estimator of α when the data are i.i.d.; the dashed line draws the density of the ML estimator when the data are generated by GARCH (1, 1) and are filtered; finally, the dotted line draws the density when the data are generated by GARCH (1, 1) but we treat them as i.i.d.. It can be seen that the difference between the i.i.d. and filtered GARCH data is very small, hence the effect from pre-filtering has little effect on the copula estimation. However, note that if the data is GARCH but we treat them as i.i.d., then $\hat{\alpha}$ is biased to the right a little bit[7]. Intuitively, when the data is conditional heteroskedastic, the clustering of large volatilities leads to underestimate of the degree of dependence.

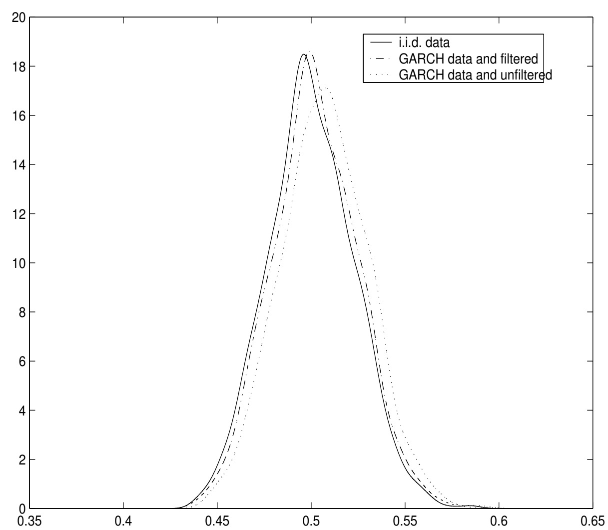

Figure 3: The distribution of $\hat{\alpha}$ when the data are i.i.d. or GARCH (Solid line: i.i.d. data; Dashed line: GARCH data and filtered; Dotted line: GARCH data treated as i.i.d.)

Finally, to implement the ML estimation of the mixture model, we adopt a popular approach: the EM algorithm (Dempster *et al.* (1977), Wu (1983), Redner and Walker (1984)). The EM algorithm is originally designed to solve ML problem with incomplete data. In a mixture model, we can treat the information that each observation is drawn from which distribution as missing data.

In summary, to estimate the mixed copula between two data series $\{X_t\}_{t=1}^n$ and $\{Y_t\}_{t=1}^n$, we first filter the original data with a GARCH model and compute the empirical CDFs of each filtered series assuming independence. Then we compute the log-likelihood function of the mixed copula with u and v replaced by the empirical CDFs $\hat{F}_X(x)$ and $\hat{F}_Y(y)$ respectively. An EM algorithm is used to implement the optimizations.

[7] In this experiment, the mean of the three cases are 0.5000 (i.i.d.), 0.5022 (filtered), 0.5070 (GARCH but unfiltered)

4 The Empirical Results

4.1 The Data

The dataset contains four stock market indices: S&P 500 (US), FTSE 100 (UK), Nikkei 225 (Japan) and Hang Seng (Hong Kong). The data is of monthly frequency and the time period is from January 1970 to September 2003, giving a total sample size of 404. The correlation matrix between the four markets are given in Table 1. It can be seen that the dependence between US and UK markets are strongest, and that between Japan and Hong Kong markets are weakest among the six pairs.

	FTSE	Nikkei	Hang Seng
S&P	0.6365	0.4176	0.4299
FTSE		0.3647	0.4505
Nikkei			0.2938

Table 1: The Linear Correlation Coefficients across Four Markets

As we summarized in the previous section, before estimating the mixture models, we first filter the data with a GARCH model. Then we compute the empirical CDF of each series. In Figure 4, the left graph plots the kernel density of filtered S&P monthly returns (in solid line) and the corresponding normal density (in dashed line). It can be seen that this univariate distribution has thicker tails and negatively skewed[8]. The right graph plots the empirical CDF of the series (in solid line) and the normal CDF (in dashed line). These features need to be considered if a full parametric approach is adopted in estimating the mixture copula. Since we use a semiparametric approach, we will only use the empirical CDFs of each series. Hence the the deviation from normal has no effect to our following analysis.

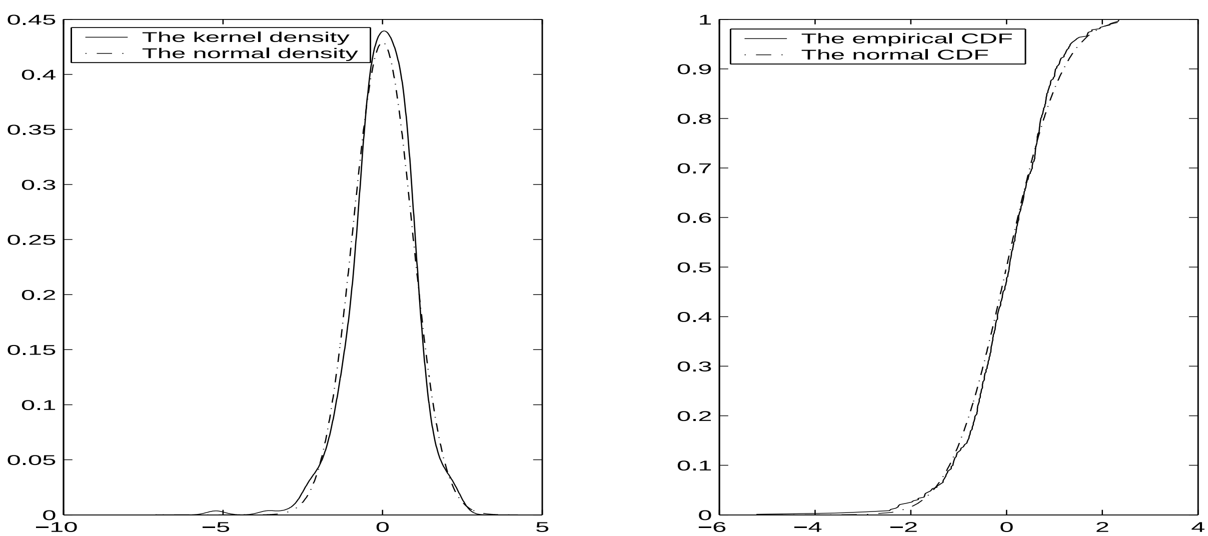

Figure 4: The kernel density and empirical CDF of filtered S&P 500 returns compared with the Normal PDF/CDF

[8]There is also a large literature documenting and discussing negative skewness in financial return series, see Chen, Hong and Stein (2001), Harvey and Siddique (2000)

4.2 The Estimates

In estimating the mixture model, to balance the goodness of fit and model parsimony, we start with the mixture of thee distributions, and then if the weight on one component is less than 0.1 or if the estimate of the association parameter in that component is close to that implies independence, we will delete this component and reestimate the mixture of the remaining distribution(s). In doing so, we could capture the key feature of the dependence while keeping the model simple.

Table 2 reports the estimates of the model[9]. Remarks: First, there is no pair which takes positive weight on the Gumbel copula, implying that the stock markets we consider does not have right tail dependence. In other words, the tendency that two markets boom together is at least not stronger than that is implied from a Gaussian dependence structure. Second, among the six pairs, Gaussian takes unit weight in the FTSE-Nikkei pair, but takes no weight or a very small weight in all other pairs. It is the GS copula that best describes the feature of dependence structure in our dataset.

	S&P, FTSE	S&P, Nikkei	S&P, Hang Seng	FTSE, Nikkei	FTSE, Hang Seng	Nikkei, Hang Seng
Gaussian ($\hat{\rho}$)	0.8227 W: 0.16			0.3507		
	(0.13)			(0.04)		
GS ($\hat{\beta}$)	0.6231 W: 0.84	0.7452	0.7239		0.7434	0.7994
	(0.04)	(0.03)	(0.03)		(0.03)	(0.03)

Table 2: The Estimates of the Mixture Copula

The findings corroborate the propositions in Longin and Solnik (2001), who computes the correlation of exceedences on both tails. They found that the correlation on the right tail is similar with that is implied from bivariate normal distribution; while the correlation of exceedences on the left tails is much more than that is implied from a bivariate normal distribution. So they draw the conclusion that it is a bear market, rather than volatility, that is the driving force increasing dependence across international equity markets.

This asymmetry is very hard to be explained by classical financial economics. According to traditional theories on financial markets, across markets co-movements are explained to be caused by some common factors. So the upward and downward movements we observe in different stock markets are due to the movements in some observable or unobservable factors that affect all markets. However, this hypothesis cannot explain why these common factors have "stronger" effects in pulling all markets down than pushing all markets up. Also, Cutler *et al.* (1989) found that large movements in prices are not associated with arrival of information.

To explain this asymmetry, one hypothesis is that investors are more sensitive to bad news than good news in other markets. When a crash take place in another market, people tend to take cautious action in the domestic market as well. And these cautious transactions may really pull the market down. On the contrary, people may not pay much attention to good news from other markets. Shiller (2001) proposed that the news media have prominent effects not only to the general public, but also to investment professionals. This findings help to support this hypothesis as the news media plays an important role in informing investors about foreign markets.

[9] In the table, if there is no weight estimate 'W', then either Gaussian or Gumbel Survival takes unit weight. The number in parenthesis are standard errors of the association parameters.

4.3 Goodness of Fit

To evaluate the performance of an estimate of dependence, usually a contingency table is used. For example, let x and y be two series and we rearrange these two series to obtain x' and y', so that $x'(t) < x'(s)$ and $y'(t) < y'(s)$ for $t < s$. Then we divide each series evenly into k parts, and we have a k by k table. Here the number k is picked based on a trade-off: we want to have enough observations in each cell, and we need enough number of groups to test contingent dependence.

Let G denotes the table and $G(i, j)$ be the cell in the ith row and the jth column, and let x'_i and y'_j denote lower bound for cell $A(i, j)$, then an observation of $(x'(t), y'(t))$ belongs to $G(i, j)$ if $x'_i < x'(t) < x'_{i+1}$ and $y'_j < y'(t) < y'_{j+1}$.

To show how this table reflects dependence, if x and y are perfectly positively correlated, then most observations should lie on the principal diagonal; if x and y are independent, then the number of observations in each cell should be approximately the same; and if x and y are perfectly negatively correlated, then most observations should lie on the diagonal connecting the upper-right corner and the lower-left corner.

Let $A_{i,j}$ and $B_{i,j}$ denote the number of observed and predicted frequencies in cell (i, j) respectively. Pearson χ^2 statistic is

$$M = \sum_i^k \sum_j^k \frac{(A_{i,j} - B_{i,j})^2}{B_{i,j}}.$$

M follows χ^2 distribution with $(k-1)^2$ degree of freedom. In practice, cells that include very few observations are usually pooled together. Suppose there are q number of cells that have been pooled together, and there are p number of parameters in the model, then the degree of freedom is $(k-1)^2 - p - (q-1)$.

Table 3 presents the contingency tables and the χ^2 statistics for all the six pairs. For each pair, the left table counted the frequencies from the data, while the right table counted the frequencies computed from the estimated model. We choose $k = 7$. Take the S&P-FTSE pair as example. The upper-left corner (cell (1, 1)) in the left column is 30, telling that out of 404 observations, there are 30 occurrences when both S&P and FTSE returns lie in their respective lowest 14th percentiles (1/7th quantile). While the cell (1, 1) in the right column is 31, telling that the model predicts 31 occurrences of these events. Take another example, the lower-left corner (cell (7, 1)) reads 1 in the left table and in the right table, telling that there is one occurrence, both from the sample and from the model, when S&P return is in its lowest 14th percentiles while FTSE return is in its highest 14th percentile.

The asymmetry in the data is now quite obvious by comparing cell (1, 1) (both markets are in their lowest 14th percentiles) and cell (7, 7) (both markets are in their highest 14th percentiles) in each table, except in the FTSE-Nikkei pair.

The χ^2 statistics and the 95% critical values corresponding to the degree of freedom in that statistics are given in the last line for each pair. For all the six pairs, the χ^2 statistics lie below their critical values.

We will need a 3-dimensional graph to plot the whole joint distributions. Note that the difference between each dependence function is quite obvious on the diagonal of $u = v$, as we plot in the lower row in Figure 1. Therefore in Figure 5, we plot the observed frequencies (in solid line) and model predicted frequencies (in dashed line) on the diagonal of the contingency table for each pair. For example, for the S&P-FTSE pair, the observed numbers on the principle diagonal are [30, 17, 13, 12, 13, 11, 23]. Divided by the sample size 404, we have frequencies [0.074, 0.042, 0.032, 0.030, 0.032, 0.027, 0.057], and these numbers are drawn in solid line in Figure 5 for that pair. From

Pair	Observed frequencies							Model Predicted frequencies						
S&P, FTSE	30	14	7	1	2	1	3	31	12	6	4	2	2	1
	14	17	10	4	2	7	3	12	15	11	8	5	4	2
	4	8	13	12	14	4	3	6	11	12	10	8	6	4
	3	5	16	12	8	8	6	4	8	10	11	10	9	6
	2	7	7	11	13	7	11	2	5	8	10	11	11	9
	4	4	5	12	12	11	9	2	4	6	9	11	13	13
	1	2	0	6	7	19	23	1	2	4	6	9	13	23
χ^2 statistics (df)	36.92 (26), c.v. 38.88													
S&P, Nikkei	26	13	9	2	3	3	2	23	11	7	5	4	4	3
	10	9	12	9	8	4	5	11	11	10	8	7	6	5
	4	12	9	11	8	6	8	7	10	10	9	8	7	6
	2	12	7	7	10	12	8	5	8	9	9	9	9	8
	5	7	7	9	6	15	9	4	7	8	9	10	10	9
	6	2	8	12	11	10	8	4	6	7	9	10	11	12
	5	2	6	8	12	7	18	3	5	6	8	9	12	15
χ^2 statistics (df)	30.7139 (34), c.v. 48.60													
S&P, Hang Seng	29	10	8	1	5	4	1	25	11	7	5	4	3	2
	10	7	15	9	6	6	4	11	12	10	8	7	6	4
	5	14	7	9	7	8	8	7	10	10	9	8	7	6
	6	14	7	10	6	6	9	5	8	9	10	9	9	8
	3	4	10	12	11	7	11	4	7	8	9	10	10	9
	3	5	6	8	10	13	12	3	6	7	9	10	11	12
	2	3	5	9	13	13	13	2	4	6	8	9	12	16
χ^2 statistics (df)	28.7710 (32), c.v. 46.19													
FTSE, Nikkei	22	10	8	8	4	1	5	17	12	9	8	6	4	3
	9	15	7	6	8	7	5	12	10	9	9	7	6	4
	6	10	10	8	11	9	4	9	9	9	9	8	7	6
	7	4	13	6	12	8	8	8	9	9	9	9	9	8
	5	4	7	13	7	11	11	6	7	8	9	9	9	9
	5	8	8	12	8	12	4	4	6	7	9	9	10	12
	4	6	5	5	8	9	21	3	4	6	8	9	12	17
χ^2 statistics (df)	38.7602 (34), c.v. 48.60													
FTSE, Hang Seng	23	10	8	3	8	5	1	24	11	7	5	4	4	3
	13	9	9	9	4	7	6	11	12	10	8	7	6	5
	7	12	9	12	7	7	4	7	10	10	9	8	7	6
	3	10	6	11	12	11	5	5	8	9	9	9	9	8
	6	7	10	8	7	11	9	4	7	8	9	10	10	9
	2	5	9	8	11	8	14	4	6	7	9	10	11	12
	4	4	7	7	9	8	19	3	5	6	8	9	12	16
χ^2 statistics (df)	24.1888 (34), c.v. 48.60													
Nikkei, Hang Seng	23	8	9	6	5	2	5	20	10	8	6	5	4	4
	12	11	14	5	6	7	2	10	11	9	8	7	6	5
	7	9	7	8	11	9	7	8	9	9	9	8	8	7
	4	9	7	9	7	10	12	6	8	9	9	9	9	8
	5	7	8	11	8	10	9	5	7	8	9	9	9	9
	2	8	7	7	11	8	14	4	6	8	9	9	10	11
	5	5	6	12	10	11	9	4	5	7	8	9	11	14
χ^2 statistics (df)	23.2146 (36), c.v. 51.00													

Table 3: Goodness of Fit: Observed Frequencies and Model Predicted Frequencies

Figure 5, we see that the model fits the data very well in most cases, and the L shape is very obvious, especially for the S&P-Hang Seng pair and Nikkei-Hang Seng pair.

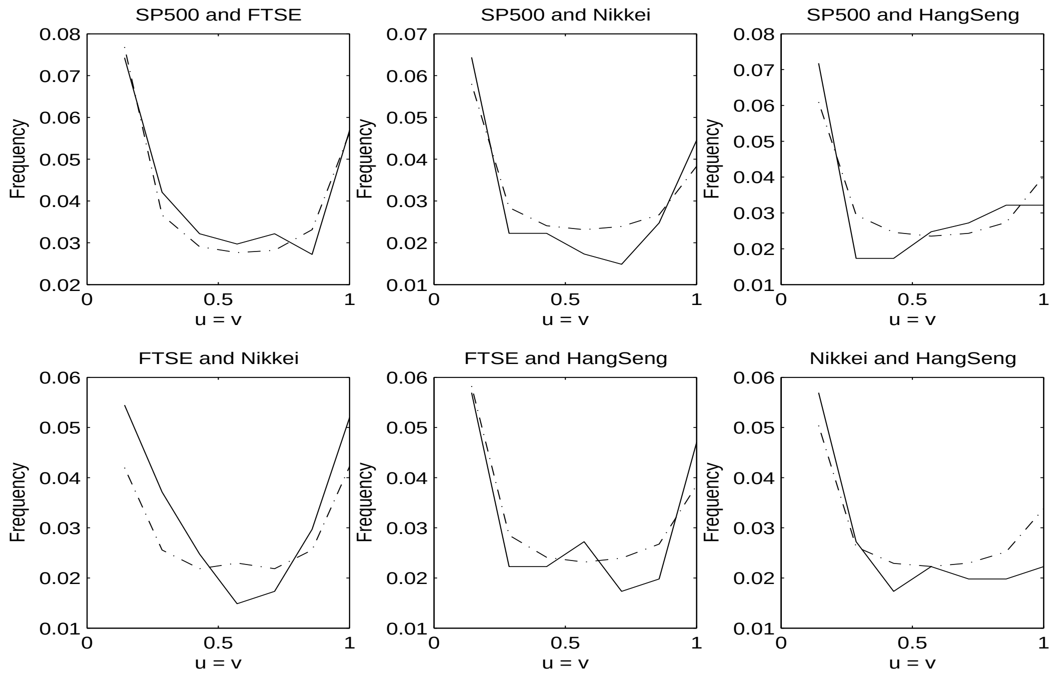

Figure 5: The observed and model predicted frequencies on the diagonal (Solid line: observed from the data; Dashed line: predicted from the model)

4.4 Application of the Model

A straightforward application of a mixed copula model is to build a joint distribution. From our results, we see that the dependence structure between financial markets displays various shapes, L shape, U shape or a mixture of them. Also, the marginal distributions themselves are skewed to the left with thick tails (see Figure 4). Hence it is not easy to build a joint distribution that is able to capture all these features at the same time. Take an example of the S&P-FTSE pair. Using the estimated copula, we can write the joint distribution as

$$\begin{aligned} H(X, Y; \gamma_x, \gamma_y) &= 0.1613 C_{\text{Gaussian}}(F_X(x; \gamma_x), F_Y(y; \gamma_y); 0.8227) \\ &\quad + 0.8387 C_{\text{GS}}(F_X(x; \gamma_x), F_Y(y; \gamma_y); 0.6230), \end{aligned}$$

where γ_x and γ_y denotes the parameters in the marginal distribution of X and Y respectively. Now the mixed copula gives the dependence structure between the two markets, while the marginals can then be specified to generate returns with certain features.

Compared with a joint Gaussian approach, which is widely assumed in multivariate studies, a mixed copula model is able to provide more realistic description of the data generating process. The implication from the estimation results to economics and finance can be summarized from the following three aspects.

First, in portfolio choice, traditional theory based on mean-variance analysis implies that investor can benefit from diversification by investing in assets with lower correlations. In our data set, the correlation between US and UK markets is about 0.64 while the correlation between US and Hong Kong markets is about 0.43. Hence mean-variance analysis tells that for US investors, Hong Kong market is a much better choice than UK market to hedge against downside movements in US domestic market. But this is not exactly true. If we define market crash as when the return is in its lowest 14th percentile, then we see that the number of occurrences that both US and UK stock market crash is 30, and that both US and Hong Kong market crash is 29. Hence when US market is bad, the probability that Hong Kong is bad is just about the same as the probability that the UK market is bad, despite the fact that the correlation coefficient between Hong Kong and US is 0.2 lower than that between UK and US.

If we simply assume joint Gaussian in the example of US and Hong Kong stock markets, the probability that both markets are in their lowest 14th percentiles is 0.048, or about 19 occurrences among 404 observations. This is much smaller than the actual number, 29. Hence in this case, using Gaussian assumption in asset allocation and risk management is very dangerous. In comparison, the copula model predicts about 25 occurrences of this event. Although still below the actual number, it is much closer to the realized number than under the Gaussian assumption.

Second, a mixed copula model is a convenient tool in risk management. Embrechts *et al.* (2003) is a general inference about using copulas in risk management. Below is a numerical example using the dataset and estimation results in the present paper to show that the copula approach provides more realistic valuation of the market risk than joint Gaussian distribution. In recent years, VaR has become a standard tool in risk management. VaR measures the loss that will happen with probability p (usually 5% or 1%) over a time period. For example, given a stationary distribution F on the return, the VaR at 1% level is $F^{-1}(0.01)$.

In the literature, computation of a portfolio VaR is largely dependent on Gaussian assumption and the emphasis is on estimation of the covariance matrix. For illustration purposes, assume a portfolio which puts equal weight in two markets and let $Z = X + Y$, where X and Y denote filtered monthly return of two stock market price indices. To compute VaR at probability p for Z using a copula C, we are looking for the threshold z for which $P(Z < z) = p$. We can find z from the following equation

$$F_Z(z) = \int_{-\infty}^{\infty} c(F_X(x), F_Y(z - x)) f_X(x) f_Y(z - x) dx = p. \tag{14}$$

We are only interested in comparing results under different dependence structures, so we also assume that the marginal distributions of X and Y are normal. Table 4 presents the results for VaR at 1% and 5% probability. Compard to the empirical VaR, the joint Gaussian approach underestimate the risk in all cases and the mixed copula approach gives a much more realistic estimate of the risk. In some cases, such as for the FTSE-Hang Seng pair and the Nikkei-Hang Seng pair, the value at risk at both 1% and 5% probability computed using the copula approach is very close to the empirical number.

Finally, the estimated dependence structure provides important information for asset pricing models. In this paper, we apply the copula approach to measure the association between international aggregate stock markets. We can also apply the model to measure the co-movements between an individual asset or a portfolio with the aggregate market. In the traditional CAPM model, this co-movement is measured by beta, which is the covariance of the asset returns with the market returns divided by the variance of the market returns. Ang *et al.* (2002) find that stocks with larger downside correlations with the market tend to have higher expected returns, given other factors controlled. In this paper, we summarize this co-movement in an estimated mixed copula function, which provides more detailed description of dependence than conditional correlations. We believe

	S&P, FTSE	S&P, Nikkei	S&P, Hang Seng	FTSE, Nikkei	FTSE, Hang Seng	Nikkei, Hang Seng
Gaussian VaR 1%	-4.0588	-3.6808	-3.7999	-3.7420	-3.9603	-3.6515
Mixed Copula VaR 1%	-4.5037	-5.0611	-4.7432	-4.2188	-4.4292	-4.4661
Empirical VaR 1%	-4.8856	-4.5195	-4.3874	-5.0046	-4.5953	-4.5254
Gaussian VaR 5%	-2.8676	-2.6025	-2.6868	-2.6458	-2.8001	-2.5818
Mixed Copula VaR 5%	-2.8872	-2.9710	-2.9609	-2.9413	-2.9006	-2.8532
Empirical VaR 5%	-2.9595	-3.0136	-2.8203	-2.9486	-2.8055	-2.8333

Table 4: Computed Value at Risk at 1% and 5% probability

the information about dependence shape should also be priced.

For example, suppose asset A and B have the same beta with the market, but when the market is booming, A goes up higher than B; and when market is crashing, A goes down less than B. In other words, A has a J shape dependence with the market and B has an L shape dependence with the market. Then, risk averse investors will prefer A to B, if all other things are the same. In other words, there is another risk: 'shape risk'. And investors will request compensation for holding assets whose shapes are not preferred, such as asset B in this example. This is a simple example for illustrative purpose and a rigorous theoretical discussion will be left for future work.

5 Conclusion

The estimation of dependence is very important in economics and finance. Our proposal is to use a mixed copula model to empirically measure cross-market dependence. In a mixed copula, the degree of dependence is carried via the association parameters, and the shape of the dependence is summarized by the weight on each individual copula function. We use semiparametric methods to estimate the model and obtain quite satisfactory goodness of fit to the data in our applications. Based on the data of four stock markets, we found that the dependence is asymmetric and display left tail dependence, implying that the markets are more likely to crash together than to boom together. We emphasize the finding that pairs that have lower correlation, such as S&P-Hang Seng, have almost the same probability to crash together as pairs that have higher correlation coefficient, such as S&P-FTSE. We expect that a mixture copula model could serve as a new tool for empirical modelings in various fields of economics and finance, including risk management and asset pricing.

References

Ang, A. and G. Bekaert (2002) International asset allocation with regime shifts, *Review of Financial Studies*, 15: 1137-1187.

Ang, A. and J. Chen (2002) Asymmetric correlations of equity portfolios, *Review of Financial Studies*, 63: 443-494.

Ang, A, J. Chen and Y. Xing (2002) Downside correlation and expected stock returns, Working paper, Columbia Business School, 2002

Bae K. H., G. A. Karolyi and R. M. Stulz (2003) A New Approach to Measuring Financial Market Contagion, forthcoming, *Review of Financial Studies*

Bouye, E., V. Durrleman, A. Nikeghbali, G. Riboulet and T. Roncalli (2000) Copulas for finance a reading guide and some applications, Working paper, Credit Lyonnais.

Boyer, B. H. and M. S. Gibson and M. Loretan (1999) Pitfalls in tests for changes in correlations, *International Finance Discussion Paper No. 597*, Board of Governors of the Federal Reserve System.

Chen, J., H. Hong and J. C. Stein (2001) Forecasting crashes: Trading volume, past returns and conditional skewness in stock prices, *Journal of Financial Economics*, 61(3): 345-381.

Cutler, D. M., J. M. Poterba and L. H. Summers (1989) What moves stock prices? *Journal of Portfolio Management*, 15: 4-12.

Dempster, A.P., N.M. Laird and D.B. Rubin (1977) Maximum likelihood estimation from incomplete data via the EM algorithm (with discussion). *Journal of the Royal Statistical Society.* Series B, 39: 1-38.

Embrechts P., F. Lindskog, and A. McNeil (2003) Modeling dependence with copulas and applications to risk management, in *Handbook of Heavy Tailed Distributions in Finance*, ed. S. Rachev, Elsevier, Chapter 8, pp. 329-384.

Embrechts, P., A. Mcneil and D. Straumann (2002) Correlation and dependence in risk management: properties and pitfalls, in *Risk Management: Value at Risk and Beyond*, ed. M.A.H. Dempster, Cambridge University Press, Cambridge, pp. 176-223.

Engle, R. (2002) Dynamic conditional correlation: A simple class of multivariate GARCH models, forthcoming in *Journal of Business and Economic Studies.*

Forbes K. and R. Rigobon (2002) No Contagion, Only Interdependence: Measuring Stock Market Co-movements, *Journal of Finance*, 57 (5): 2223-2261.

Genest, C. and L. Rivest (1993) Statistical inference procedures for bivariate Archimedean copulas, *Journal of the American Statistical Association*, 88: 1034-1043.

Genest, C., K. Ghoudi and L. Rivest (1995) A semiparametric estimation procedure of dependence parameters in multivariate families of distributions, *Biometrika*, 82:543-552.

Harvey, C. R. and A. Siddique (2000) Conditional skewness in asset pricing tests, *Journal of Finance*, 55: 1263-1295.

Joe H., (1997) Multivariate models and dependence concepts, Volumn 37 in *Monographs on statistics and applied probability.*

Karolyi, G. A. and R. M. Stulz (1996) Why do markets move together? An investigation of US-Japan stock return comovements? *Journal of Finance*, 51(3): 951-986

Li, D. X. (2000) On default correlation: a copula function approach, *Journal of Fixed Income*, 9: 43-54.

Longin, F. and B. Solnik (2001) Extreme correlation of international equity markets, *Journal of Finance*, 56(2): 649-676.

Mashal R. and A. Zeevi (2002) Beyond correlation: extreme co-movements between financial assets, Working Paper, Columbia Business School.

Nelson, R. B. (1999) *An Introduction to Copulas*, Springer, New York.

Redner R. and H. Walker (1984) Mixture densities, maximum likelihood and the EM algorithm, *SIAM Review*, 26(2).

Shiller, R. J (2001) Bubbles, Human Judgment, and Expert Opinion, Cowles Foundation Discussion Paper No. 1303.

Sklar, A. (1959) Fonctions de repartition a n dimentional et leurs marges. *Publications de l'Institut de Statistique de l'Universite de Paris*, 8: 229-231.

Wu C. F. J. (1983) On the convergence properties of the EM algorithm, *The annals of Statistics*, 11(1): 95-103.