Dependence Modeling with Archimedean Copulas

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1 Introduction

A *copula* is a function that joins or "couples" a bivariate distribution function $H(x, y)$ to its one-dimensional marginal distribution functions $F(x)$ and $G(y)$—defined implicitly by the relationship $H(x, y) = C(F(x), G(y))$. Equivalently, a copula is a bivariate distribution function with uniform $(0, 1)$ margins. For a formal treatment of copulas and their properties, see the monographs by Hutchinson and Lai (1990), Joe (1997), and Nelsen (1999), and conference proceedings edited by Beneš and Štěpán (1997), Cuadras et al. (2002), Dall'Aglio et al. (1991), Dhaene et al. (2003), and Rüschendorf et al. (1996).

The importance of copulas in statistical modeling is partially explained in Sklar's Theorem (1959): *Let H be a two-dimensional distribution function with marginal distribution functions F and G. Then there exists a copula C such that H(x,y) = C(F(x),G(y)). Conversely, for any distribution functions F and G and any copula C, the function H defined above is a two-dimensional distribution function with marginals F and G. Furthermore, if F and G are continuous (as we shall assume), C is unique.*

Sklar's theorem also applies to survival functions. Let X, Y be continuous random variables with survival functions $\overline{F}(x) = \Pr(X > x)$ and $\overline{G}(x) = \Pr(Y > y)$, and joint survival function $\overline{H}(x) = \Pr(X > x, Y > y)$. The function $\hat{C}$ which couples $\overline{H}$ to $\overline{F}$ and $\overline{G}$ via $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(x))$ is called the *survival copula* of X and Y. Furthermore, $\hat{C}$ is a copula, and $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$, where C is the (ordinary) copula of X and Y.

If one has a collection of copulas, then using Sklar's theorem, one can construct bivariate distributions with arbitrary marginal distributions. Thus, for the purposes of statistical modeling, it is desirable to have a large collection of copulas at one's disposal. A great many examples of copulas can be found in the literature, most are members of families with one or more real parameters. An important class of such families are the Archimedean copulas.

2 Archimedean Copulas

An *Archimedean* copula is a function C from $[0,1]^2$ to $[0,1]$ given by $C(u,v) = \phi^{[-1]}(\phi(u)+\phi(v))$, where ϕ (the *generator* of C) is a continuous strictly decreasing convex function from $[0,1]$ to $[0,\infty]$ such that $\phi(1)=0$, and where $\phi^{[-1]}$ denotes the "pseudo-inverse" of ϕ: $\phi^{[-1]}(t) = \phi^{-1}(t)$ for $t \in [0,\phi(0)]$, and $\phi^{[-1]}(t) = 0$ for $t \geq \phi(0)$. When $\phi(0) = \infty$, ϕ and C are said to be *strict* (and $\phi^{[-1]} = \phi^{-1}$); when $\phi(0) < \infty$, ϕ and C are *non-strict*. Furthermore, $C(u,v) > 0$ on $(0,1]^2$ if and only if C is strict.

The term Archimedean for these copulas arises as follows. One form of the Archimedean property states that if u and v are real numbers in $(0,1)$, then there exists an integer n such that $u^n < v$. Given a copula C, define "C-multiplication" $\circ$ by $u \circ v = C(u,v)$. Then u^n is defined recursively by $u^2 = u \circ u$ and $u^n = u^{n-1} \circ u$. Archimedean copulas satisfy the above Archimedean property with respect to the C-multiplication $\circ$. [Note: non-Archimedean copulas also have this property as long as $C(u,u) < u$ for u in $(0,1)$.] The term "Archimedean copula" first appeared in the statistical literature in two papers by Genest and Mackay (1986ab), although the word Archimedean was employed by Ling in 1965 for Archimedean t-norms (every Archimedean copula is also an Archimedean t-norm). Archimedean copulas also appear in Schweizer and Sklar (1983) but without the name.

Archimedean copulas are widely used in applications due to their simple form, a variety of dependence structures, and other"nice" properties. For example, most but not all extend to higher dimensions via the associativity property [C is associative if $C(C(u,v),w) = C(u,C(v,w))$]. A collection of twenty-two one-parameter families of Archimedean copulas can be found in Table 4.1 of Nelsen (1999).

3 Simulation

In this section we present two algorithms to generate an observation (u,v) from an Archimedean copula C with generator ϕ.

ALGORITHM I:
1. Generate two independent uniform $(0,1)$ variates s and t;
2. Set $w = K^{(-1)}(t)$,where $K(t) = t - \phi(t)/\phi'(t^+)$ $[K^{(-1)}(t) = \sup\{x|K(t) \leq x\}]$;
3. Set $w = \phi^{[-1]}\Big(s\phi(w)\Big)$ and $v = \phi^{[-1]}\Big((1-s)\phi(w)\Big)$.

Algorithm I is a consequence of the fact that if U and V are uniform random variables with an Archimedean copula C, then $W = C(U,V)$ and $S = \phi(U)/(\phi(U)+\phi(V))$ are independent, S is uniform $(0,1)$, and the distribution function of W is K (Genest and Rivest, 1993).

ALGORITHM II:
1. Generate two independent uniform $(0,1)$ variates u and t;
2. Set $w = \phi'^{(-1)}\Big(\phi'(u)/t\Big)$;

3. Set $v = \phi^{[-1]}\Big(\phi(w) - \phi(u)\Big)$.

Algorithm II is the "conditional distribution function" method, where $v = c_u^{(-1)}(t)$ for $c_u(t) = \partial C(u,v)/\partial u = P[V \le v | U = u]$.

In the talk we will present scatterplots for samples from several families of Archimedean copulas to illustrate the variety of dependence structures present.

4 The zero set and the zero curve

The *zero set* of a copula C is $Z(C) = \Big\{(u,v) \in [0,1]^2 \,|C(u,v) = 0\Big\}$. For C Archimedean with generator ϕ, $C(u,v) = 0$ is equivalent to $\phi(u) + \phi(v) \ge \phi(0)$. When C is strict $[\phi(0) = \infty]$, $Z(C)$ consists of the two line segments $\{0\} \times [0,1]$ and $[0,1] \times \{0\}$. When C is non-strict $[\phi(0) < \infty]$ $Z(C)$ can have positive area, and the zero set is the portion of $[0,1]^2$ below the *zero curve* $\phi(u) + \phi(v) = \phi(0)$.

Example 1. Let $\phi(t) = (1-t)^2$. Then the zero curve of the Archimedean copula generated by ϕ is one-quarter of the circle centered at $(1,1)$ with radius 1; and the zero set is the portion of the square $[0,1]^2$ bounded by the quarter circle and the two axes. □

The probability mass assigned to the zero curve depends on the ordinate and the one-sided derivative of the generator at 0:

THEOREM 1. *If C is a non-strict Archimedean copula with generator ϕ, then the C-measure of the zero curve $\phi(u) + \phi(v) = \phi(0)$ is*

$$-\frac{\phi(0)}{\phi'(0^+)},$$

and hence equal to 0 whenever $\phi'(0^+) = -\infty$.

5 Two examples of an Archimedean dependence structure

Example 2. In the Proceedings of the First Brazilian Conference on Statistical Modelling in Insurance and Finance, we considered the following problem (also see Schmitz, 2004). Let $\{X_1, X_2, \cdots, X_n\}$ be a set of independent and identically distributed continuous random variables with distribution function F, and let $X_{(1)} = \min\{X_i\}$ and $X_{(n)} = \max\{X_i\}$. Let $C_{1,n}$ denote the copula of $X_{(1)}$ and $X_{(n)}$. For convenience, we first found the joint distribution function H^* and copula C^* of $-X_{(1)}$ and $X_{(n)}$, rather than $X_{(1)}$ and $X_{(n)}$:

$$H^*(s,t) = \begin{cases} [F(t) - F(-s)]^n, & -s \le t, \\ 0, & -s > t, \end{cases}$$

and to find C^*, we "inverted:" $C^*(u,v) = H^*(G^{(-1)}(u), F_n^{(-1)}(v))$, where G denotes the distribution function of $-X_{(1)}$ and F_n the distribution function of $X_{(n)}$. Hence $C^*(u,v) = \left[\max\left(u^{1/n} + v^{1/n} - 1, 0\right)\right]^n$, a member of the Clayton family (4.2.1) in

Table 4.1 of Nelsen (1999) (with $\theta = -1/n$). Since $X_{(1)}$ is a decreasing function of $-X_{(1)}$, the copula $C_{1,n}$ of $X_{(1)}$ and $X_{(n)}$ is

$$C_{1,n}(u,v) = v - C^*(1-u,v) = v - \left[\max\left((1-u)^{1/n} + v^{1/n} - 1, 0\right)\right]^n.$$

Although $X_{(1)}$ and $X_{(n)}$ are clearly not independent ($C_{1,n}(u,v) \neq uv$), they are asymptotically independent since $\lim_{n\to\infty} C_{1,n}(u,v) = uv$. □

Example 3. Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a vector of continuous nonnegative random variables ("lifetimes") with a Schur-constant joint survival function $P(\mathbf{X} > x) = S(x_1 + \cdots + x_n)$ for an appropriate function S. The lifetimes X_i are exchangeable, lower-dimensional margins are Schur-constant, and the X_i satisfy an *indifference-to-aging* property: two residual lifetimes $(X_i - x_i)$ and $(X_j - x_j)$ of different ages x_i and x_j have the same conditional distribution: $P(X_i - x_i > t | \mathbf{X} > \mathbf{x}) = P(X_j - x_j > t | \mathbf{X} > \mathbf{x})$ for any $\mathbf{x}$ in $[0,\infty)^n$.

THEOREM 2. *The components of* $\mathbf{X}$ *with a Schur-constant survival function are independent if and only if they are exponentially distributed.*

Thus the indifference-to-aging property is a generalization of the lack-of-memory property. The following lemma and theorem exhibit the relationship between Schur-constant survival functions and Archimedean copulas.

LEMMA 3. *Let S be a continuous (univariate) survival function. Then $S(x+y)$ is a bivariate survival function if and only if S is convex.*

THEOREM 4. *Let X and Y be lifetimes with a Schur-constant survival function $P(X > x, Y > y) = S(x+y)$. Then X and Y possess an Archimedean survival copula whose generator is the inverse of S.*

A converse holds, and hence there is a one-to-one correspondence between Schur-constant survival models (modulo a scale parameter) and Archimedean copulas. For example:

Pareto survival functions	$\Longleftrightarrow$	Clayton copulas
Weibull survival functions	$\Longleftrightarrow$	Gumbel-Hougaard copulas
Gompertz survival functions	$\Longleftrightarrow$	Gumbel-Barnett copulas, etc.

For details and the extension to n dimensions, see (Nelsen, 2005). □

6 Multi-parameter families

In many instances, a single parameter does not provide sufficient flexibility for modeling purposes. In this section we discuss methods to add a parameter.

Let ϕ be a generator, and define $\phi_{\alpha,1}(t) = \phi(t^\alpha)$ and $\phi_{1,\beta}(t) = [\phi(t)]^\beta$. Then:

i) $\phi_{1,\beta}$ is a generator for all $\beta \geq 1$;

ii) $\phi_{\alpha,1}$ is a generator for all α in $(0,1]$;

iii) if ϕ is twice differentiable and $t\phi'(t)$ is nondecreasing on (0,1), then $\phi_{\alpha,1}$ is a generator for all $\alpha > 0$.

The set $\{\phi_{\alpha,1}\}$ is the *interior power family* of generators associated with ϕ and $\{\phi_{1,\beta}\}$ is the *exterior power family* of generators associated with ϕ (Oakes, 1994).

Example 4. (Fang et al., 2000): $\phi_\theta(t) = \ln([1-\theta(1-t]/t)$ generates the Ali-Mikhail-Haq family [(4.2.3 in Table 4.1 of Nelsen (1999)]; and since $t\phi'_\phi(t)$ is nondecreasing for θ in $[0,1]$, the interior power family of copulas associated with ϕ_θ is, for u, v, θ in $[0,1]$, $\alpha > 0$:

$$C_{\theta;\alpha,1}(u,v) = \frac{uv}{[1-\theta(1-u^{1/\alpha})(1-v^{1/\alpha})]^\alpha}. \ \square$$

For another application, let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be i.i.d. pairs of random variables with a common Archimedean copula C (generator ϕ) and let $C_{(n)}$ denote the copula of the component-wise maxima $X_{(n)} = \max\{X_i\}$ and $Y_{(n)} = \max\{Y_i\}$. Then

$$C_{(n)}(u,v) = C^n(u^{1/n}, v^{1/n}) = \left[\phi^{[-1]}\left(\phi(u^{1/n}) + \phi(v^{1/n})\right)\right]^n.$$

The generator of $C_{(n)}$ is $\phi_{1/n,1}(t) = \phi(t^{1/n})$, and thus the copula of the component-wise maxima is a member of the interior power family generated by ϕ.

Finally, we note that we can create a two-parameter family of generators from a single generator ϕ: $\phi_{\alpha,\beta}(t) = [\phi(t^\alpha)]^\beta$ for appropriate values of α and β.

7 Measures of association

The two most commonly employed measures of association are Spearmans rho (ρ) and Kendalls tau (τ). If X and Y are continuous random variables with copula C, then

$$\rho_{X,Y} = \rho_C = \begin{cases} 12\int_0^1\int_0^1 uv dC(u,v) - 3, \\ 12\int_0^1\int_0^1 C(u,v) dudv - 3, \end{cases}$$

and

$$\tau_{X,Y} = \tau_C = \begin{cases} 4\int_0^1\int_0^1 C(u,v) dC(u,v) - 1, \\ 1 - 4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v} dudv. \end{cases}$$

For Archimedean copulas, there does not appear to be a simple expression for Spearman's ρ in terms of the generator ϕ; however, for Kendall's τ we have (Genest and Mackay, 1986ab; Joe, 1997)

$$\tau_C = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}\, dt = 1 - 4\int_0^\infty u\left[\frac{d}{du}\phi^{[-1]}(u)\right]^2 du.$$

Example 5. Let C_θ be a Clayton copula, i.e., $C_\theta(u,v) = \left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-1/\theta}$, generated by $\phi_\theta(t) = \left(t^{-\theta} - 1\right)/\theta$ for $\theta \geq -1$. Then

$$\frac{\phi_\theta(t)}{\phi'_\theta(t)} = \frac{t^{\theta+1} - t}{\theta} \quad (\theta \neq 0), \quad \frac{\phi_0(t)}{\phi'_0(t)} = t\ln t,$$

and hence $\tau_\theta = \theta/(\theta+2)$. In Example 2, with $X_{(1)} = \min\{X_i\}$ and $X_{(n)} = \max\{X_i\}$, the copula of $-X_{(1)}$ and $X_{(n)}$ is a Clayton copula with $\theta = -1/n$, and hence Kendall's tau for $-X_{(1)}$ and $X_{(n)}$ is $-1/(2n-1)$. Thus Kendall's tau for $X_{(1)}$ and $X_{(n)}$ is $1/(2n-1)$. □

8 Tail Dependence

Let X and Y be continuous random variables with distribution functions F and G, respectively. The *upper tail dependence parameter* λ_U is the limit (if it exists) of the conditional probability that Y is greater than the $100t$-th percentile of G given that X is greater than the $100t$-th percentile of F as t approaches 1, i.e.

$$\lambda_U = \lim_{t\to 1^-} P\left[Y > G^{(-1)}(t)\,\middle|\,X > F^{(-1)}(t)\right].$$

Similarly, the *lower tail dependence parameter* λ_L is the limit (if it exists) of the conditional probability that Y is less than or equal to the $100t$-th percentile of G given that X is less than or equal to the $100t$-th percentile of F as t approaches 0, i.e.

$$\lambda_L = \lim_{t\to 0^+} P\left[Y \le G^{(-1)}(t)\,\middle|\,X \le F^{(-1)}(t)\right]$$

These parameters are nonparametric and depend only on the copula C of X and Y, and its *diagonal section* $\delta_C(t) = C(t,t)$:

Theorem 5. $\lambda_L = \lim_{t\to 0^+} \frac{C(t,t)}{t} = \delta'_C(0^+)$ *and* $\lambda_U = 2 - \lim_{t\to 1^-} \frac{1-C(t,t)}{1-t} = 2 - \delta'_C(1^-)$.

When C is Archimedean with generator ϕ, we have

Corollary 6. $$\lambda_L = \lim_{t\to 0^+} \frac{\varphi^{[-1]}(2\varphi(t))}{t} = \lim_{x\to\infty} \frac{\varphi^{[-1]}(2x)}{\varphi^{[-1]}(x)}$$

and

$$\lambda_U = 2 - \lim_{t\to 1^-} \frac{1-\varphi^{[-1]}(2\varphi(t))}{1-t} = 2 - \lim_{x\to 0^+} \frac{1-\varphi^{[-1]}(2x)}{1-\varphi^{[-1]}(x)}.$$

The parameters λ_U and λ_L can be evaluated for the twenty-two families of Archimedean copulas in Table 4.1 of Nelsen (1999):

Family	λ_L	λ_U
3, 5, 7-11, 13, 17, 22	0	0
2, 4, 6, 15, 21	0	$2-2^{1/\theta}$
18	0	1
$1(\theta \ge 0)$	$2^{-1/\theta}$	0
12	$2^{-1/\theta}$	$2-2^{1/\theta}$
16	1/2	0
14	1/2	$2-2^{1/\theta}$
19, 20	1	0

Note: The values of the parameters can be different for a limiting case. For example, the copula denoted $\Pi/(\Sigma-\Pi)$ has $\lambda_L = 1/2$ although it is a limiting case in families (4.2.3), (4.2.8) and (4.2.19); and M has $\lambda_U = 1$ although it is a limiting case in families (4.2.1), (4.2.5), (4.2.13), etc.

Examining the above table raises the following question: Can we find an Archimedean copula with *arbitrary* (positive) values of λ_L and λ_U? The answer is yes, and the method uses interior and exterior power families. Let C be Archimedean with generator ϕ and upper and lower tail dependence parameters λ_U and λ_L. Let $C_{\alpha,1}$ and $C_{1,\beta}$ denote the copulas generated by $\phi_{\alpha,1}(t) = \phi(t^\alpha)$ and $\phi_{1,\beta}(t) = [\phi(t)]^\beta$, respectively. Then the upper and lower tail dependence parameters of $C_{\alpha,1}$ are λ_U and $\lambda_L^{1/\alpha}$, respectively, and the upper and lower tail dependence parameters of $C_{1,\beta}$ are $2 - (2 - \lambda_U)^{1/\beta}$ and $\lambda_L^{1/\beta}$, respectively.

Example 6. The copula $\Pi/(\Sigma - \Pi)$ is generated by $\phi(t) = (1/t) - 1$ with $\lambda_U = 0$ and $\lambda_L = 1/2$. Hence $\phi_{\alpha,\beta}(t) = (t^{-\alpha} - 1)^\beta$ generates the two-parameter family (for $\alpha > 0$, $\beta \geq 1$)

$$C_{\alpha,\beta}(u, v) = \left\{ \left[(u^{-\alpha} - 1)^\beta + (v^{-\alpha} - 1)^\beta \right]^{1/\beta} + 1 \right\}^{-1/\alpha}.$$

Here $\lambda_{U;\alpha,\beta} = 2 - 2^{1/\beta}$ and $\lambda_{L;\alpha,\beta} = 2^{-1/\alpha\beta}$. If λ_U^* and λ_L^* are the desired values of the tail dependence parameters, we solve $\lambda_U^* = 2 - 2^{1/\beta}, \lambda_L^* = 2^{-1/\alpha\beta}$ for α and β, to obtain $\alpha = -\ln(2 - \lambda_U^*/ \ln \lambda_L^*$ and $\beta = \ln 2/\ln(2 - \lambda_U^*)$. □

9 Multivariate Archimedean copulas

First we introduce some notation, points in n-space will be denoted by $\mathbf{u} = (u_1, u_2, \cdots, u_n)$; $\mathbf{a} \leq \mathbf{b}$ means $a_k \leq b_k$ for all k; and for $\mathbf{a} \leq \mathbf{b}$, $[\mathbf{a}, \mathbf{b}]$ denotes the *n-box* $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$; and the *vertices* of $[\mathbf{a}, \mathbf{b}]$ are points $\mathbf{c}$ such that each c_k is equal to either a_k or b_k.

An n-dimensional copula (or n-copula) is a function $C : \mathbf{I}^n \to \mathbf{I}$ such that
(i) for every $\mathbf{u}$ in $\mathbf{I}^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except u_k,
(ii) C is *n-increasing*: for every $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{I}^n$ such that $\mathbf{a} \leq \mathbf{b}$, the *C-volume* of $[\mathbf{a}, \mathbf{b}]$ is $V_C([\mathbf{a}, \mathbf{b}]) = \Sigma \text{sgn}(\mathbf{c}) C(\mathbf{c}) \geq 0$, where the sum is over the vertices $\mathbf{c}$ of $[\mathbf{a}, \mathbf{b}]$ and $\text{sgn}(\mathbf{c}) = 1$ if $c_k = a_k$ for an even number of ks, and -1 if $c_k = a_k$ for an odd number of ks.

In general, constructing n-copulas is difficult. One of the most important open problems concerning copulas is the *compatibility problem*. For $n = 3$, it is: Given three 2-copulas C_1, C_2, and C_3, construct a 3-copula C with C_1, C_2, and C_3 as its 2-dimensional margins, i.e., such that $C(1, v, w) = C_1(v, w), C(u, 1, w) = C_2(u, w)$ and $C(u, v, 1) = C_3(u, v)$.

However, the associativity property enables us to often (but not always) extend Archimedean copulas to higher dimensions. Let ϕ be a strict generator, and C the function from $\mathbf{I}^n$ to $\mathbf{I}$ given by

$$C\left(u_1, u_2, \cdots, u_n\right) = \phi^{-1}\Big(\phi(u_1) + \phi(u_2) + \cdots + \phi(u_n)\Big).$$

Then for every $n \geq 2$, C satisfies the boundary conditions for a copula, and will satisfy the n-increasing condition if and only if ϕ^{-1} is *completely monotone* (Kimberling, 1974):

$$(-1)^k \frac{d^k}{dt^k}\phi^{-1}(t) \geq 0 \text{ for all } t \in (0,\infty) \text{ and } k = 0, 1, \cdots$$

Many of the families in Table 4.1 of Nelsen (1999) have ϕ^{-1} completely monotone for the portion of the parameter range for which $C_\theta(u,v) \geq uv$: families (4.2.n) for $n =$ 1, 3, 4, 5, 6, 12, 13, 14, and 19.

10 Some open questions

1. There are numerous statistical arguments that are used to justify the assumption of normality. Are there similar arguments that can be used to justify the assumption that the copula of two random variables is Archimedean?
2. Archimedean copulas are *associative*, i.e., $C(C(u,v),w) = C(u,C(v,w))$. What does associativity mean statistically?
3. If an Archimedean copula is appropriate for a given data set, are there statistical procedures for choosing a particular family (i.e., for choosing the generator)?
4. How do Archimedean families differ statistically? For example:

(a) The *Frank* family is the only radially symmetric family, i.e., the copula C and the survival copula $\hat{C}$ coincide.

(b) The *Gumbel-Hougaard* family is the only max-stable family (and hence the only family of Archimedean extreme value copulas), i.e., $\left(X_{(n)}, Y_{(n)}\right)$ and (X,Y) have the same copula (Genest and Rivest, 1989).

(c) The *Clayton* family is the only "truncation invariant" family: If the copula of U and V is C, then for any u_0, v_0 in $(0,1)$, the copula of the conditional random variables $U|U \leq u_0$, $V|V \leq v_0$ is also C(Oakes, 2004).

5. Why does $\Pi/(\Sigma - \Pi)$ appear in so many families (7 of 22) of Table 4.1 in Nelsen (1999)?

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