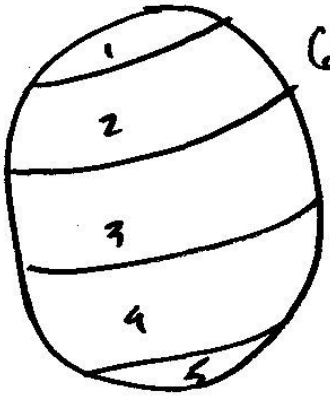
Stratified sampling

Stat 576
4-22-25



Collect SRSWOR within each stratum

Optimally, we want homogeneity within, heterogeneity between

4 big questions:
1. How many strata?
2. What should the boundaries be? Demarcation
3. How large should n be?
4. What portion of the sample should come from each stratum? Allocation

Notation:

H = # of strata

y_{hj} = j^{th} item sampled in stratum h

N = population size

N_h = " " of stratum h

n = sample size

n_h = " " from stratum h

$\bar{Y}$ = population mean

$\bar{Y}_h$ = " " of stratum h

S_h^2 = modified population variance for stratum h

t_h = population total for stratum h
$= N_h \bar{Y}_h$

$\bar{y}_h$ = sample mean for stratum h

s_h^2 = " variance " " "

Know: $E[\bar{y}_h] = \bar{Y}_h$, $V[\bar{y}_h] = \frac{S_h^2}{n_h}\left(1 - \frac{n_h}{N_h}\right)$,

$\hat{V}[\bar{y}_h] = \frac{s_h^2}{n_h}\left(1 - \frac{n_h}{N_h}\right)$

Goal: Estimate $\bar{Y}$

Write $\bar{Y} = \frac{\sum_h N_h \bar{Y}_h}{N} = \sum_{h=1}^{H} \frac{N_h}{N} \bar{Y}_h$

An unbiased estimator of $\bar{Y}$ is

$$\bar{y}_{str} = \frac{1}{N} \sum_{h=1}^{H} N_h \bar{y}_h$$

$$V[\bar{y}_{str}] = \frac{1}{N^2} \sum_{h=1}^{H} N_h^2 V[\bar{y}_h]$$

$$= \frac{1}{N^2} \sum_{h=1}^{H} N_h^2 \frac{S_h^2}{n_h}\left(1 - \frac{n_h}{N_h}\right)$$

$$\hat{V}[\bar{y}_{str}] = \frac{1}{N^2} \sum_{h=1}^{H} N_h^2 \frac{s_h^2}{n_h}\left(1 - \frac{n_h}{N_h}\right)$$ (5)

Allocation:

What type of allocation would yield a self-weighting sample? (i.e. $\bar{y}_{str} = \bar{y}$)

$$\bar{y}_{str} = \frac{\sum_{h=1}^{H} N_h \bar{y}_h}{N} = \frac{1}{N} \sum_{h=1}^{H} N_h \sum_{j=1}^{n_h} y_{hj} \frac{1}{n_h}$$

$$= \sum_{h=1}^{H} \sum_{j=1}^{n_h} \frac{N_h}{N n_h} y_{hj}$$

To force $\bar{y}_{str} = \bar{y}$, we need $\frac{N_h}{N n_h} = \frac{1}{n}$ (6)

So $n_h = \frac{N_h}{N} n$, which is <u>proportional allocation</u>

ANOVA decomposition:

$$SS_{TOT} = \sum_{h=1}^{H} \sum_{j=1}^{N_h} (y_{hj} - \bar{y})^2$$

$$= \sum_{h=1}^{H} \sum_{j=1}^{N_h} \left(\underline{y_{hj} - \bar{y}_h} + \underline{\bar{y}_h - \bar{y}}\right)^2$$

(7)

$$= \underbrace{\sum_h \sum_j (Y_{hj} - \bar{Y}_h)^2}_{①} + \underbrace{\sum_h \sum_j (\bar{Y}_h - \bar{Y})^2}_{②} + 2 \underbrace{\sum_h \sum_j (Y_{hj} - \bar{Y}_h)(\bar{Y}_h - \bar{Y})}_{③}$$

$$① = \sum_{h=1}^{H} (N_h - 1) S_h^2 = \text{"within strata" sum of squares}$$

$$② = \sum_{h=1}^{H} N_h (\bar{Y}_h - \bar{Y})^2 = \text{"between strata" sum of squares}$$

(8)

$$③ = 2 \sum_{h=1}^{H} \left[(\bar{Y}_h - \bar{Y}) \underbrace{\sum_{j=1}^{N_h} (Y_{hj} - \bar{Y}_h)}_{\sum_{j=1}^{N_h} Y_{hj} - N_h \bar{Y}_h} \right]$$

$$\text{So } SS_{TOT} = SS_{within} + SS_{between}$$

Since $V[\bar{y}_{str}]$ only involves SS_{within}, ideally SS_{within} should be small & $SS_{between}$ large

Optimal allocation

Assume we have a setup of C_0

& a per-item cost of C_h in stratum h

Then the total cost of the sample is

$$C = C_0 + \sum_{h=1}^{H} n_h C_h$$

Also, $V[\bar{y}_{str}] = \frac{1}{N^2} \sum_{h=1}^{H} N_h^2 \frac{S_h^2}{n_h} \left(1 - \frac{n_h}{N_h}\right)$

$$= \frac{1}{N^2} \sum_{h=1}^{H} N_h^2 \frac{S_h^2}{n_h} - \frac{1}{N^2} \sum_{h=1}^{H} N_h^2 \frac{S_h^2}{n_h} \frac{n_h}{N_h} \quad (10)$$

We will minimize

$$\left(\sum_{h=1}^{H} n_h C_h\right)\left(\frac{1}{N^2} \sum_{h=1}^{H} N_h^2 \frac{S_h^2}{n_h}\right)$$

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$

$$(\vec{u} \cdot \vec{v})^2 = |\vec{u}|^2 |\vec{v}|^2 \cos^2\theta \leq |\vec{u}|^2 |\vec{v}|^2$$

Cauchy-Schwarz Inequality

(11)

Let $u_h = \frac{N_h}{N} \frac{S_h}{\sqrt{n_h}}$ and $v_h = \sqrt{n_h c_h}$

Now we are minimizing $\left(\sum_{h=1}^{H} v_h^2\right)\left(\sum_{h=1}^{H} k_h^2\right)$

$= |\vec{v}|^2 \; |\vec{u}|^2$

C-S says that this is minimized when $\vec{u} = k\vec{v}$

that is, $\frac{N_h}{N} \frac{S_h}{\sqrt{n_h}} = k\sqrt{n_h c_h}$

(12)

$$n_h = \frac{N_h}{N} \frac{S_h}{k\sqrt{c_h}} \quad \text{But } k = ?$$

$$n = \sum_{h=1}^{H} n_h = \sum_{h=1}^{H} \frac{N_h}{N} \frac{S_h}{k\sqrt{c_h}}$$

$$\Rightarrow k = \frac{1}{n} \sum_{h=1}^{H} \frac{N_h}{N} \frac{S_h}{\sqrt{c_h}}$$

$$\therefore n_h = \frac{N_h S_h}{N\sqrt{c_h}} \frac{n}{\sum_{h=1}^{H} \frac{N_h}{N} \frac{S_h}{\sqrt{c_h}}}$$

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$$n_h = \frac{\frac{N_h S_h}{\sqrt{C_h}}}{\sum_{h=1}^{H} \frac{N_h S_h}{\sqrt{C_h}}} \cdot n$$

This is called optimal allocation

What if the cost per item is the same in all strata?

$$n_h = \frac{N_h S_h}{\sum_{h=1}^{H} N_h S_h} \cdot n$$

This is called Neyman allocation

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What if the population variances are the same in all strata?

$$n_h = \frac{N_h}{\sum_{h=1}^{H} N_h} \cdot n = \frac{N_h}{N} \cdot n$$, which is proportional allocation

What if the strata are all the same size?

$$n_h = \frac{\frac{N}{H} \cdot n}{N} = \frac{n}{H}$$, which is equal allocation