

Confidence Intervals

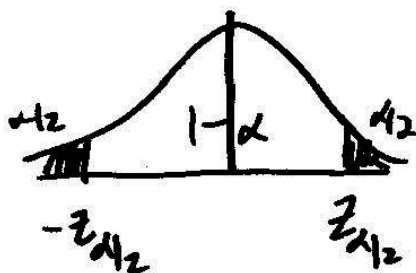
Stat 576
4-8-25

SRS WR:

①

By the Central Limit Theorem,

$$\frac{\bar{y} - \bar{Y}}{\sqrt{V(\bar{y})}} \xrightarrow{D} N(0,1) \text{ as } n \rightarrow \infty$$



$$P\left(-z_{\alpha/2} < \frac{\bar{y} - \bar{Y}}{\sqrt{V(\bar{y})}} < z_{\alpha/2}\right) \approx 1 - \alpha$$

$$\bar{y} - z_{\alpha/2} \sqrt{V(\bar{y})} < \bar{Y} < \bar{y} + z_{\alpha/2} \sqrt{V(\bar{y})}$$

this is a $(1-\alpha)$ conf. I.

$$\begin{aligned} \bar{y} \pm z_{\alpha/2} \sqrt{V(\bar{y})} &= \bar{y} \pm z_{\alpha/2} \sqrt{\frac{\sigma^2}{n}} \\ &= \bar{y} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \end{aligned}$$

point estimator margin of error

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Usually, σ^2 must be estimated by s^2 ,
resulting in $\boxed{\bar{y} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}}$ for large n

RSWOR:

③

Hájek (1960) proved an extension to the C.L.T.

For large $n, N, N-n$,

$$\frac{\bar{y} - \bar{Y}}{\sqrt{V(\bar{y})}} \approx N(0,1)$$

Result $\bar{y} \pm z_{\alpha/2} \sqrt{\frac{S^2}{n}}$ is a $(1-\alpha)100\%$ C.I. for $\bar{Y}$

Also s^2 can estimate S^2, s_u

$$\boxed{\bar{y} \pm z_{\alpha/2} \frac{s}{\sqrt{n}} \sqrt{1 - \frac{n}{N}}}$$

Suppose that we specify a margin of

④

error, E . Find the necessary sample size.

WR: Set $z_{\alpha/2} \frac{s}{\sqrt{n}} = E$ & solve for n

$$n = \left(\frac{z_{\alpha/2} s}{E} \right)^2$$

WOR: Set $z_{\alpha/2} \frac{s}{\sqrt{n}} \sqrt{1 - \frac{n}{N}} = E$

$$z^2 \frac{s^2}{n} \left(1 - \frac{n}{N}\right) = E^2$$

$$\frac{\sum z^2 s^2}{n} - \frac{\sum z^2 \bar{s}^2}{N} = E^2 \quad (5)$$

$$\sum z^2 s^2 = n \left(\frac{\sum z^2 \bar{s}^2}{N} + E^2 \right)$$

$$n = \frac{\sum z^2 s^2}{\frac{\sum z^2 \bar{s}^2}{N} + E^2} = \frac{\frac{\sum z^2 s^2}{E^2}}{1 + \frac{1}{N} \frac{\sum z^2 \bar{s}^2}{E^2}}$$

$$n = \frac{n_0}{1 + \frac{n_0}{N}}, \quad \text{where } n_0 \text{ is the WR solution}$$

Example: In a sample of 100 items selected from a population of 1000 items, we find $\bar{y} = 63$ and $s = 10$. (6)

Find a 95% C.I. for $\bar{Y}$

$$\text{WR: } 63 \pm 1.96 \frac{10}{\sqrt{100}} = 63 \pm 1.96$$

$$\text{WOR: } 63 \pm 1.96 \frac{10}{\sqrt{100}} \sqrt{1 - \frac{100}{1000}} = 63 \pm 1.86$$

Suppose the desired E is 1

(7)

$$\text{WR: } n = \frac{z^2 S^2}{E^2} = \frac{1.96^2 10^2}{1^2} = 384.16 \rightarrow \underline{385}$$

$$\text{WOR: } n = \frac{n_0}{1 + \frac{n_0}{N}} = \frac{384.16}{1 + \frac{384.16}{1000}} = 277.6 \rightarrow \underline{278}$$

Proportions Assume $Y_j = \begin{cases} 1 \\ 0 \end{cases}$ for $j=1, \dots, N$

$$\text{Then } \bar{Y} = \frac{1}{N} \sum_{j=1}^N Y_j = \frac{\#1\text{s}}{N} = p = \text{population proportion}$$

We know that $\bar{y}$ is an unbiased estimator of $\bar{Y}$ (8)

$$\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i = \frac{\#1\text{s}}{n} = \hat{p} = \text{sample proportion}$$

$$\begin{aligned} \sigma^2 &= \frac{1}{N} \sum_{j=1}^N Y_j^2 - (\bar{Y})^2 \\ &= \frac{1}{N} \sum_{j=1}^N Y_j - p^2 = p - p^2 = p(1-p) \\ &= pq \end{aligned}$$

$$S^2 = \frac{N}{N-1} \sigma^2 = \frac{N}{N-1} pq$$

$$s^2 = \frac{1}{n-1} \left[\sum_{i=1}^n y_i^2 - n \bar{y}^2 \right]$$

⑨

$$= \frac{1}{n-1} \left[\underbrace{\sum_{i=1}^n y_i}_{n\bar{y}} - n \hat{p}^2 \right] = \frac{1}{n-1} n (\hat{p} - \hat{p}^2)$$

$$= \frac{n}{n-1} \hat{p} (1 - \hat{p})$$

$$= \frac{1}{n-1} \hat{p} \hat{q}$$

C.I. for p , SRSWR

$$\hat{p} \pm z_{\alpha/2} \frac{\sqrt{\frac{n}{n-1} \hat{p} \hat{q}}}{\sqrt{n}}$$

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p} \hat{q}}{n-1}}$$

⑩

C.I. for p , SRSWOR

$$\hat{p} \pm z_{\alpha/2} \frac{\sqrt{\frac{1}{n-1} \hat{p} \hat{q}}}{\sqrt{n}} \sqrt{1 - \frac{n}{N}}$$

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p} \hat{q}}{n-1}} \sqrt{1 - \frac{1}{N}}$$

Sample size determination

(11)

WR: Set $z_{\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n-1}} = E$

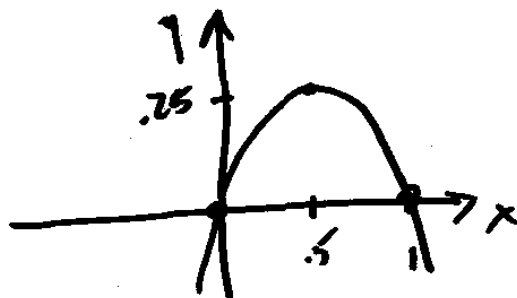
$$n = 1 + \frac{z^2 \hat{p} \hat{q}}{E^2} = n_0$$

WOR: Set $z_{\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n-1}} \sqrt{1 - \frac{1}{n}} = E$

$$n = \frac{n_0}{1 + \frac{n_0 - 1}{n}}$$

$$y = x(1-x) = x - x^2$$

(12)



If a pilot study is not feasible, use $\hat{p}\hat{q} = .25$ to overestimate n