Stat 526
4-24-25

We had:

$$S_{xy} - \sum_i \sum_j \hat{\tau}_i (x_{ij} - \bar{x}_{..}) - \hat{\beta} S_{xx} = 0 \qquad (1)$$

$$\vdots \quad \hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$$

$$S_{xy} - \sum_i \sum_j \left[\bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})\right](x_{ij} - \bar{x}_{..}) - \hat{\beta} S_{xx} = 0$$

$$S_{xy} - \sum_i \left(\left[\bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})\right] \underbrace{\sum_j (x_{ij} - \bar{x}_{..})}_{x_{i.} - n\bar{x}_{..}}\right) - \hat{\beta} S_{xx} = 0$$

(2)

$$S_{xy} - n \sum_i \left(\left[\bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})\right](\bar{x}_{i.} - \bar{x}_{..})\right) - \hat{\beta} S_{xx} = 0$$

$$S_{xy} - T_{xy} + \hat{\beta} T_{xx} - \hat{\beta} S_{xx} = 0$$

$$\therefore \hat{\beta} = \frac{S_{xy} - T_{xy}}{S_{xx} - T_{xx}} = \frac{E_{xy}}{E_{xx}}$$

Results: $\hat{\mu} = \bar{y}_{..}$, $\hat{\beta} = \dfrac{E_{xy}}{E_{xx}}$,

$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$$

To test $H_0: \tau_i \equiv 0$, use the additional sum of squares F test.

Full: $$y_{ij} = \mu + \tau_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$$

Reduced: $$y_{ij} = \mu + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$$

Full: $$SSE = \sum_i \sum_j \left[y_{ij} - \cancel{\bar{y}_{..}} - \left(\bar{y}_{i.} - \cancel{\bar{y}_{..}} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \right. \right. \\ \left. - \hat{\beta}(x_{ij} - \bar{x}_{..}) \right]^2$$

$$= \sum_i \sum_j \left[y_{ij} - \bar{y}_{i.} - \hat{\beta}(x_{ij} - \bar{x}_{i.}) \right]^2$$

$$SSE = E_{yy} + \hat{\beta}^2 E_{xx} - 2\hat{\beta} E_{xy}$$

$$= E_{yy} + \left(\frac{E_{xy}}{E_{xx}}\right)^2 E_{xx} - 2\left(\frac{E_{xy}}{E_{xx}}\right) E_{xy}$$

$$\underset{\text{(full)}}{SSE} = E_{yy} - \frac{E_{xy}^2}{E_{xx}}$$

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Full model ANOVA

Source	SS	df
Model	$T_{yy} + \frac{E_{xy}^2}{E_{xx}}$	a
Error	$E_{yy} - \frac{E_{xy}^2}{E_{xx}}$	$an-(a+1)$
Total	S_{yy}	$an-1$

Reduced: $$y_{ij} = \underbrace{(\mu - \beta \bar{x}_{..})}_{\beta_0} + \underbrace{\beta}_{\beta_1} x_{ij} + \varepsilon_{ij}$$

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From 564, we know: $$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

$$\hat{\beta}_0 = \bar{y}_{..} - \hat{\beta}_1 \bar{x}_{..}$$

So $$\hat{\beta} = \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

$$\hat{\mu} - \hat{\beta}\bar{x}_{..} \stackrel{\text{set}}{=} \bar{y}_{..} - \hat{\beta}_1 \bar{x}_{..}$$

$$\therefore \hat{\mu} = \bar{y}_{..}$$

Also, $SS_E = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$ $\quad df_E = an - 2$

(7)

Additional sum of squares F test:

$$F = \frac{\left(\dfrac{SSE_{red} - SSE_{full}}{dfE_{red} - dfE_{full}}\right)}{MSE_{full}}$$

$$= \frac{\left[\dfrac{S_{yy} - \dfrac{S_{xy}^2}{S_{xx}} - \left(E_{yy} - \dfrac{E_{xy}^2}{E_{xx}}\right)}{an - 2 - (an - (a+1))}\right]}{MSE_{full}}$$

(8)

$$F = \frac{\left[\dfrac{T_{yy} - \dfrac{S_{xy}^2}{S_{xx}} + \dfrac{E_{xy}^2}{E_{xx}}}{a-1}\right]}{MSE_{full}}$$

$df = a-1,\ an - (a+1)$

Usual ANOCOVA layout:

Source	SS	df
Reg	S_{xr}^2/S_{xx}	1
TRT	$T_{yy} - \frac{S_{xy}^2}{S_{xx}} + \frac{E_{xy}^2}{E_{xx}}$	$a-1$
ERR	$E_{yy} - \frac{E_{xy}^2}{E_{xx}}$	$an-(a+1)$
TOT	S_{yy}	$an-1$

Notes:

Type I sums of squares are sequential

Type II " " " " conditional

Matrix version

$i = 1, \dots, a$

$j = 1, \dots, n$

$N = an$

$$y_{ij} = \mu + \tau_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$$

$$\underbrace{\begin{bmatrix} y_{11} \\ \vdots \\ y_{1n} \\ \hline y_{21} \\ \vdots \\ y_{2n} \\ \hline \vdots \\ \vdots \\ \hline y_{a1} \\ \vdots \\ y_{an} \end{bmatrix}_{N\times 1}}_{Y} = \underbrace{\left[\begin{array}{c|cccc|c} 1 & 1 & 0 & \cdots & 0 & x_{11} - \bar{x}_{..} \\ \vdots & \vdots & & & & \vdots \\ 1 & 1 & 0 & \cdots & 0 & x_{1n} - \bar{x}_{..} \\ \hline 1 & 0 & 1 & 0 \cdots & 0 & x_{21} - \bar{x}_{..} \\ \vdots & \vdots & & & & \vdots \\ 1 & 0 & 1 & 0 \cdots & 0 & x_{2n} - \bar{x}_{..} \\ \hline \vdots & & & \vdots & & \vdots \\ \vdots & & & \vdots & & \vdots \\ \hline 1 & -1 & & \cdots & -1 & x_{a1} - \bar{x}_{..} \\ \vdots & \vdots & & & & \vdots \\ 1 & -1 & & \cdots & -1 & x_{an} - \bar{x}_{..} \end{array}\right]_{N\times(a+1)}}_{X} \underbrace{\begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_{a-1} \\ \beta \end{bmatrix}_{a+1}}_{\vec{\beta}} + \vec{\varepsilon}_{N\times 1}$$

HW #4

14.20

14.21

14.20. An experiment is designed to study pigment dispersion in paint. Four different mixes of a particular pigment are studied. The procedure consists of preparing a particular mix and then applying that mix to a panel by three application methods (brushing, spraying, and rolling). The response measured is the percentage reflectance of pigment. Three days are required to run the experiment, and the data obtained follow. Analyze the data and draw conclusions, assuming that mixes and application methods are fixed.

		Mix			
Day	**Application Method**	**1**	**2**	**3**	**4**
	1	64.5	66.3	74.1	66.5
1	2	68.3	69.5	73.8	70.0
	3	70.3	73.1	78.0	72.3
	1	65.2	65.0	73.8	64.8
2	2	69.2	70.3	74.5	68.3
	3	71.2	72.8	79.1	71.5
	1	66.2	66.5	72.3	67.7
3	2	69.0	69.0	75.4	68.6
	3	70.8	74.2	80.1	72.4

14.21. Repeat Problem 14.20, assuming that the mixes are random and the application methods are fixed.