Stat 526
4-10-25

Satterthwaite's Approximate F test

①

Construct 2 new MS terms

$$MS' = MS_t + \dots + MS_u$$

$$MS'' = MS_v + \dots + MS_w$$

no terms in common

so that $E(MS') - E(MS'') =$ Constant times the desired variance component

For yesterday's example
Try $MS' = MS_A + MS_{ABC}$

$MS'' = MS_{AB} + MS_{AC}$

②

Satterthwaite proved that $\frac{MS'}{MS''}$ has

an approximate F distribution with p & q df,

where $p = \dfrac{(MS')^2}{\frac{MS_t^2}{df_t} + \dots + \frac{MS_u^2}{df_u}}$, $q = \dfrac{(MS'')^2}{\frac{MS_v^2}{df_v} + \dots + \frac{MS_w^2}{df_w}}$

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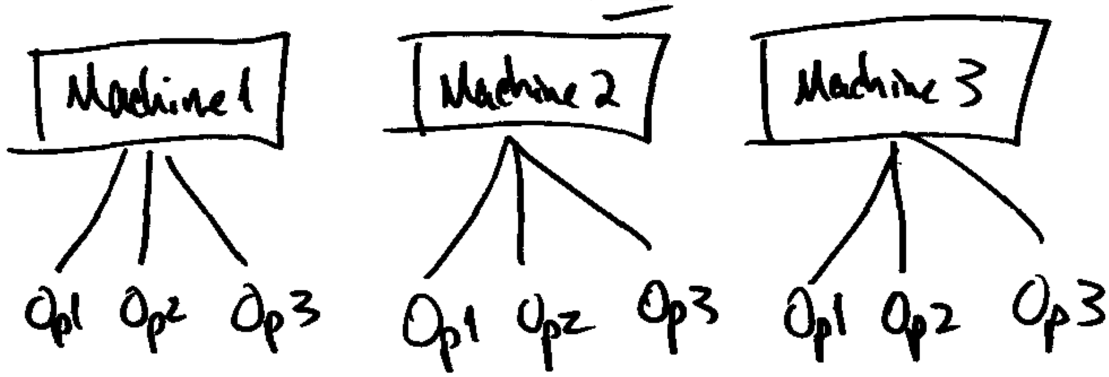
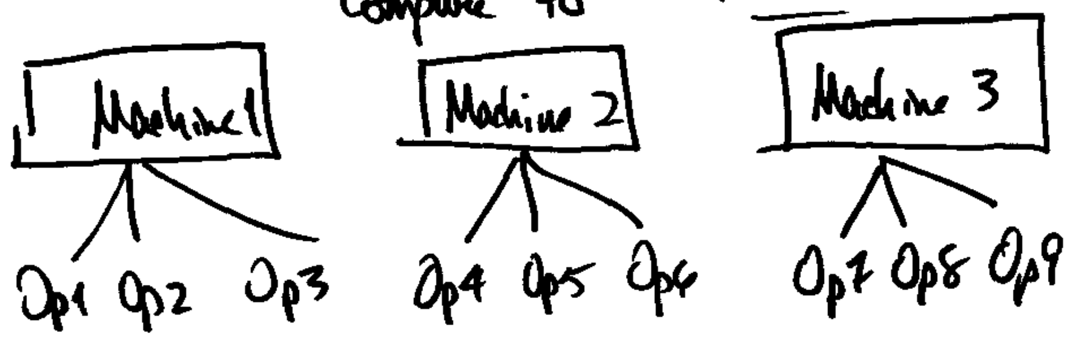
Nested designs

Crossed



Compare to

Nested



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Model for a nested design:

$$y_{ijk} = \mu + \underset{\substack{\uparrow \\ \text{machine}}}{\tau_i} + \underset{\substack{\uparrow \\ \text{operator, nested} \\ \text{within machine}}}{\beta_{j(i)}} + \varepsilon_{(ij)k}$$

Parameter estimates, assuming both effects are fixed

Assume $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_{j(i)} = 0 \quad \forall i$

$$SSE = \sum_i \sum_j \sum_k \left[y_{ijk} - (\mu + \tau_i + \beta_{j(i)}) \right]^2$$

$$\frac{\partial SSE}{\partial \mu} = \sum_i \sum_j \sum_k \cancel{2}\left[y_{ijk} - (\mu + \tau_i + \beta_{j(i)})\right](-\cancel{1}) \overset{set}{=} 0$$ ⑤

$$y_{\cdots} - N\mu - 0 - 0 = 0$$

$$\therefore \hat{\mu} = \frac{y_{\cdots}}{N} = \bar{y}_{\cdots}$$

$$\frac{\partial SSE}{\partial \tau_i} = \sum_j \sum_k \cancel{2}\left[y_{ijk} - (\mu + \tau_i + \beta_{j(i)})\right](-\cancel{1}) \overset{set}{=} 0$$

$$y_{i\cdot\cdot} - bn\mu - bn\tau_i - 0 = 0$$

$$\hat{\tau}_i = \frac{y_{i\cdot\cdot} - bn\bar{y}_{\cdots}}{bn} = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$$

⑥

$$\frac{\partial SSE}{\partial \beta_{j(i)}} = \sum_k \cancel{2}\left[y_{ijk} - (\mu + \tau_i + \beta_{j(i)})\right](-\cancel{1}) \overset{set}{=} 0$$

$$y_{ij\cdot} - n\mu - n\tau_i - n\beta_{j(i)} = 0$$

$$\hat{\beta}_{j(i)} = \frac{y_{ij\cdot} - n\bar{y}_{\cdots} - n(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})}{n}$$

$$= \bar{y}_{ij\cdot} - \cancel{\bar{y}_{\cdots}} - \bar{y}_{i\cdot\cdot} + \cancel{\bar{y}_{\cdots}}$$

$$= \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot}$$

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Compare to the crossed model:

$$\hat{\mu} = \bar{y}_{\cdots}, \quad \hat{\tau}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$$

$$\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}, \quad \widehat{\tau\beta}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}$$

SS & df, using rule from Tuesday's class

$df_A = a-1$ $\quad\quad\quad \tau_i$

$df_{B(A)} = (b-1)a = ab - a$ $\quad\quad\quad \beta_{j(i)}$

Compare to crossed: $df_A = a-1$, $df_B = b-1$,
$df_{AB} = (a-1)(b-1)$

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$$SS_A = \sum_i \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdots}^2}{N}$$

$$SS_{B(A)} = \sum_i \sum_j \frac{y_{ij\cdot}^2}{n} - \sum_i \frac{y_{i\cdot\cdot}^2}{bn}$$

<u>Example</u>: A: machine <u>fixed</u> a levels

B: operator <u>random</u> nested within machine

b operators " "

n replicates

	$\overset{a}{F}$ i	$\overset{b}{R}$ j	$\overset{n}{R}$ k	EMS
τ_i	0	b	n	$bn \frac{\sum \tau_i^2}{a-1} + n\sigma_\beta^2 + \sigma^2$
$\beta_{j(i)}$	1	1	n	$n\sigma_\beta^2 + \sigma^2$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

(9)

F tests for A: $\frac{MS_A}{MS_{B(A)}}$

B(A): $\frac{MS_{B(A)}}{MS_E}$

<u>Example</u>: y measures hardness of material (10)

2 different alloy formulations

3 different heats for each alloy

2 ingots of each alloy

2 replications

A: alloy $a=2$ fixed

B: heat $b=3$ fixed nested within A

C: ingot $c=2$ random nested with both A & B

$n=2$

(11)

	$\overset{a}{F}$ i	$\overset{b}{F}$ j	$\overset{c}{R}$ k	$\overset{n}{R}$ l	EMS
τ_i	0	3	2	2	$12\sum\tau_i^2 + 2\sigma_\gamma^2 + \sigma^2$
$\beta_{j(i)}$	1	0	2	2	$\cancel{4}\frac{\sum\sum\beta_{j(i)}^2}{\cancel{4}} + 2\sigma_\gamma^2 + \sigma^2$
$\gamma_{k(ij)}$	1	1	1	2	$2\sigma_\gamma^2 + \sigma^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

A: $MSA / MS_{C(AB)}$ df = 1, 6

B(A): $MS_{B(A)} / MS_{C(AB)}$ df = 4, 6

C(AB): $\frac{MS_{C(AB)}}{MS_E}$ df = 6, 12

(12)

Source	df	
A	1	$a-1$
B(A)	4	$(b-1)a$
C(AB)	6	$(c-1)ab$
ERR	12	$(n-1)abc$
TOT	23	$N-1$

HW #2

13.17

13.17. Consider a four-factor factorial experiment where factor A is at a levels, factor B is at b levels, factor C is at c levels, factor D is at d levels, and there are n replicates. Write down the sums of squares, the degrees of freedom, and the expected mean squares for the following cases. Assume the restricted model for all mixed models. You may use a computer package such as Minitab.

(**a**) A, B, C, and D are fixed factors.

(**b**) A, B, C, and D are random factors.

(**c**) A is fixed and B, C, and D are random.

(**d**) A and B are fixed and C and D are random.

(**e**) A, B, and C are fixed and D is random.

Do exact tests exist for all effects? If not, propose test statistics for those effects that cannot be directly tested.