Stat 566
4-1-25
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Chapter 13

Random effects

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \qquad \begin{array}{l} i = 1, \ldots, a \\ j = 1, \ldots, n \end{array}$$

$E[\varepsilon_{ij}] \equiv 0$

$V[\varepsilon_{ij}] = \sigma^2$

τ_i's are random variables

$E[\tau_i] \equiv 0$

$V[\tau_i] \equiv \sigma^2_\tau$

All of the ε_{ij}'s and τ_i's are independent

$H_0: \sigma^2_\tau = 0$

$H_1: \sigma^2_\tau > 0$

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Decomposition of sum of squares

$$\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2}_{SS_{TOT}} = \underbrace{n\sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2}_{SS_{TRT}} + \underbrace{\sum_i\sum_j (y_{ij} - \bar{y}_{i.})^2}_{SS_E}$$

Find $E[SS_{TRT}] = n\,E\left[\sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2\right]$

$$= n\,E\left[\sum_{i=1}^{a} \bar{y}_{i.}^2 - 2\bar{y}_{..}\underbrace{\sum_{i=1}^{a} \bar{y}_{i.}}_{a\bar{y}_{..}} + a\bar{y}_{..}^2\right]$$

$$= n\,E\left[\sum_{i=1}^{a} \bar{y}_{i.}^2 - a\bar{y}_{..}^2\right]$$

Let $E[\bar{y}_{i.}^2] = C_i$

Let $E[\bar{y}_{..}^2] = D$

(3)

C_i:
$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$
$$\bar{y}_{i.} = \mu + \tau_i + \bar{\varepsilon}_{i.}$$
$$\bar{y}_{i.}^2 = \mu^2 + \tau_i^2 + \bar{\varepsilon}_{i.}^2 + 2\mu\tau_i + 2\mu\bar{\varepsilon}_{i.} + 2\tau_i\bar{\varepsilon}_{i.}$$

$$C_i = E[\bar{y}_{i.}^2] = \mu^2 + \underbrace{E[\tau_i^2]}_{\substack{= \\ V[\tau_i] + (E[\tau_i])^2 \\ \sigma_\tau^2 + 0}} + \underbrace{E[\bar{\varepsilon}_{i.}^2]}_{\substack{V[\bar{\varepsilon}_{i.}] + (E[\bar{\varepsilon}_{i.}])^2 \\ \frac{\sigma^2}{n} + 0}} + 0 + 0 + 0$$

$$C_i = \mu^2 + \sigma_\tau^2 + \frac{\sigma^2}{n}$$

(4)

D:
$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$
$$\bar{y}_{..} = \mu + \bar{\tau}_{.} + \bar{\varepsilon}_{..}$$
$$\bar{y}_{..}^2 = \mu^2 + \bar{\tau}_{.}^2 + \bar{\varepsilon}_{..}^2 + 2\mu\bar{\tau}_{.} + 2\mu\bar{\varepsilon}_{..} + 2\bar{\tau}_{.}\bar{\varepsilon}_{..}$$

$$D = E[\bar{y}_{..}^2] = \mu^2 + E[\bar{\tau}_{.}^2] + E[\bar{\varepsilon}_{..}^2] + 0 + 0 + 0$$
$$= \mu^2 + \frac{\sigma_\tau^2}{a} + \frac{\sigma^2}{an}$$

Now $E[SS_{TRT}] = n\left[\sum_{i=1}^{a} C_i - aD\right]$

$$= n\left[a\left(\mu^2 + \sigma_\tau^2 + \frac{\sigma^2}{n}\right) - a\left(\mu^2 + \frac{\sigma_\tau^2}{a} + \frac{\sigma^2}{an}\right)\right]$$

$$= n\left[(a-1)\sigma_\tau^2 + \sigma^2\left(\frac{a}{n} - \frac{1}{n}\right)\right]$$

$$= (a-1)\left[n\sigma_\tau^2 + \sigma^2\right] = E\left[SS_{TRT}\right]$$

Define $MS_{TRT} = \dfrac{SS_{TRT}}{a-1}$

Then $E\left[MS_{TRT}\right] = n\sigma_\tau^2 + \sigma^2$

Under $H_0: \sigma_\tau^2 = 0$, $E\left[MS_{TRT}\right] = \sigma^2$

Recall: In the fixed effects model,

$$E\left[MS_{TRT}\right] = n\frac{\sum_{i=1}^{a}\tau_i^2}{a-1} + \sigma^2$$

Facts (require normality assumption on all τ_i's and ε_{ij}'s)

1. $\dfrac{SSE}{\sigma^2} \sim \chi^2_{N-a}$ $(N = na)$

2. $\dfrac{SS_{TRT}}{\sigma^2} \sim \chi^2_{a-1}$

3. SS_E and SS_{TRT} are independent

$\Rightarrow$ Under H_0,

$$\frac{\frac{SS_{TRT}}{\sigma^2}\Big/(a-1)}{\frac{SSE}{\sigma^2}\Big/(N-a)} = \frac{MS_{TRT}}{MS_E} \sim F_{a-1,N-a}$$

(7)

So there is no difference in the F test for
a 1-way ANOVA with fixed vs. random effects.

Suppose H_0 is rejected and you wish to estimate σ^2_τ

$$E[MS_{TRT}] = n\sigma^2_\tau + \sigma^2$$

$$E[MS_E] = \sigma^2$$

$$E\left[\frac{MS_{TRT} - MS_E}{n}\right] = \sigma^2_\tau$$

So an unbiased estimator of σ^2_τ is $\hat{\sigma}^2_\tau = \frac{MS_{TRT} - MS_E}{n}$

$\hat{\sigma}^2_\tau$ doesn't have a known distribution. (8)

Consider $\frac{\sigma^2_\tau}{\sigma^2 + \sigma^2_\tau}$

$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$

$V(y_{ij}) = \sigma^2_\tau + \sigma^2$

This is the proportion of the variance of y_{ij}
that is explained by the factor.

We know: Under H_0, $\frac{SS_{TRT}}{\sigma^2} \sim \chi^2_{a-1}$

Under either H_0 or H_1, $\frac{SS_E}{\sigma^2} \sim \chi^2_{N-a}$

Fact: Under H_1, $\frac{SS_{TRT}}{\sigma^2 + n\sigma_\tau^2} \sim \chi^2_{a-1}$

$$\frac{\frac{SS_{TRT}}{\sigma^2 + n\sigma_\tau^2} \Big/ (a-1)}{\frac{SSE}{\sigma^2} \Big/ (N-a)} = \frac{MS_{TRT}}{MS_E} \frac{\sigma^2}{\sigma^2 + n\sigma_\tau^2} \sim F_{a-1, N-a}$$

To be finished next time.