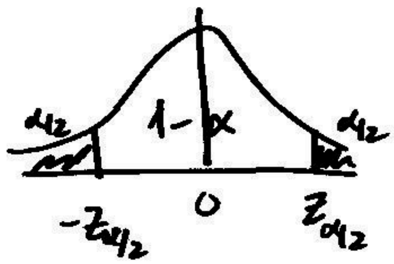
Applications of the last 2 theorems

Stat 563
5-28-26

The first theorem said (subject to regularity conditions), ①

$$\frac{\hat{\theta}_{MLE} - \theta}{\sqrt{\frac{1}{nI(\theta)}}} \xrightarrow{\mathcal{D}} N(0,1)$$

For large n



$$P\left(-z_{\alpha/2} \le \frac{\hat{\theta}_{MLE} - \theta}{\sqrt{\frac{1}{nI(\theta)}}} \le z_{\alpha/2}\right) \approx 1-\alpha$$

②

$$P\left(\hat{\theta} - z_{\alpha/2}\sqrt{\frac{1}{nI(\theta)}} \le \theta \le \hat{\theta} + z_{\alpha/2}\sqrt{\frac{1}{nI(\theta)}}\right) \approx 1-\alpha$$

So an approximate $(1-\alpha)100\%$ Conf. Int for θ

is $\quad \hat{\theta}_{MLE} \pm z_{\alpha/2}\sqrt{\frac{1}{nI(\theta)}}$

Since θ is still unknown, we must replace $I(\theta)$ with $I(\hat{\theta})$

The second theorem said (with regularity conditions)

$$-2\ln\Lambda \xrightarrow{\mathcal{D}} \chi^2_1$$

2 alternatives:

① Wald test

Under H_0, $$\frac{\hat{\theta}-\theta_0}{\sqrt{\frac{1}{nI(\theta_0)}}} \xrightarrow{\mathcal{D}} N(0,1)$$

Also $$\hat{\theta} \xrightarrow{P} \theta_0$$

④

So $$I(\hat{\theta}) \xrightarrow{P} I(\theta_0)$$

Write $$\frac{\hat{\theta}-\theta_0}{\sqrt{\frac{1}{nI(\hat{\theta})}}} = \frac{\overbrace{\frac{\hat{\theta}-\theta_0}{\sqrt{\frac{1}{nI(\theta_0)}}}}^{\xrightarrow{\mathcal{D}} N(0,1)}}{\underbrace{\sqrt{\frac{I(\theta_0)}{I(\hat{\theta})}}}_{\xrightarrow{P} 1}}$$

$$\xrightarrow{\mathcal{D}} N(0,1)$$

$$\therefore \left(\frac{\hat{\theta}-\theta_0}{\sqrt{\frac{1}{nI(\hat{\theta})}}}\right)^2 \xrightarrow{\mathcal{D}} \chi^2_1$$

⑤

② Rao score test

Consider $\left[\dfrac{\ell'(\theta_0)}{\sqrt{n\,I(\theta_0)}}\right]^2$

Recall $\dfrac{1}{\sqrt{n}}\ell'(\theta_0) = I(\theta_0)\sqrt{n}(\hat{\theta}-\theta_0) + \underbrace{R_1}_{\xrightarrow{P} 0}$

$$\frac{\ell'(\theta_0)}{\sqrt{n\,I(\theta_0)}} = \sqrt{I(\theta_0)\,n}\,(\hat{\theta}-\theta_0) + \underbrace{R_2}_{\xrightarrow{P} 0}$$

$$= \frac{\hat{\theta}-\theta_0}{\sqrt{\frac{1}{nI(\theta_0)}}} + R_2$$

⑥

So $\dfrac{\ell'(\theta_0)}{\sqrt{n\,I(\theta_0)}} \xrightarrow{\mathcal{D}} N(0,1)$

$$\left(\frac{\ell'(\theta_0)}{\sqrt{n\,I(\theta_0)}}\right)^2 \xrightarrow{\mathcal{D}} \chi^2_1$$

Example: $X_1, \dots, X_n \sim$ iid Bernoulli (p)

$H_0: p = p_0$

$H_1: p \neq p_0$

$$f(x) = p^x (1-p)^{1-x} \qquad x = 0,1$$

$$L(p) = \prod_{i=1}^{n} \left(p^{x_i} q^{1-x_i} \right)$$

$$= p^{\Sigma x_i} q^{n - \Sigma x_i}$$

$$\ell(p) = \Sigma x_i \ln p + (n - \Sigma x_i) \ln(1-p)$$

$$\ell'(p) = \Sigma x_i \frac{1}{p} + (n - \Sigma x_i) \frac{1}{1-p}(-1) \overset{\text{set}}{=} 0$$

(8)

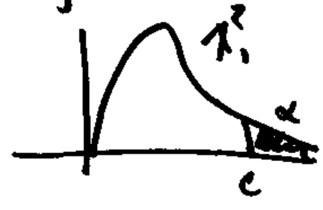
$$\hat{p} = \frac{\Sigma x_i}{n}$$

$$\Lambda = \frac{L(p_0)}{L(\hat{p})} = \frac{p_0^{\Sigma x_i} (1-p_0)^{n-\Sigma x_i}}{\left(\frac{\Sigma x_i}{n}\right)^{\Sigma x_i} \left(1 - \frac{\Sigma x_i}{n}\right)^{n - \Sigma x_i}}$$

& reject H_0 when $\Lambda \le c$

Approximate solutions:

Compute $-2 \ln \Lambda$ & reject when this $\geq c$



Wald's test: $f(x) = p^x (1-p)^{1-x}$ (9)

$$\ln f = x \ln p + (1-x)\ln(1-p)$$

$$\frac{\partial \ln f}{\partial p} = \frac{x}{p} + \frac{1-x}{1-p}(-1)$$

$$\frac{\partial^2 \ln f}{\partial p^2} = -\frac{x}{p^2} + \frac{1-x}{(1-p)^2}(-1)$$

$$I(p) = -E\left[\frac{\partial^2 \ln f}{\partial p^2}\right]$$

$$= E\left[\frac{x}{p^2} + \frac{1-x}{(1-p)^2}\right] = \frac{p}{p^2} + \frac{q}{q^2}$$

$$I(p) = \frac{1}{p} + \frac{1}{q} = \frac{q+p}{pq} = \frac{1}{pq}$$ (10)

$$\left[\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{1}{nI(\hat{\theta})}}}\right]^2 = \left[\frac{\hat{p} - p_0}{\sqrt{\frac{1}{n\frac{1}{\hat{p}\hat{q}}}}}\right]^2$$

$$= \left[\frac{\hat{p} - p_0}{\sqrt{\frac{\hat{p}\hat{q}}{n}}}\right]^2$$

Reject H_0 when this $\geq c$ as before

(11)

Rao's score test

$$\ell'(p) = \frac{\Sigma x_i}{p} - \frac{n-\Sigma x_i}{1-p}$$

$$\ell'(p_0) = \frac{\Sigma x_i}{p_0} - \frac{n-\Sigma x_i}{1-p_0} \qquad \hat{p} = \frac{\Sigma x_i}{n}$$

$$= \frac{n\hat{p}}{p_0} - \frac{n\hat{q}}{q_0}$$

$$\left(\frac{\ell'(\theta_0)}{\sqrt{n I(\theta_0)}}\right)^2 = \frac{\left(\frac{n\hat{p}}{p_0} - \frac{n\hat{q}}{q_0}\right)^2}{n \frac{1}{p_0 q_0}}$$

(12)

$$= \left(\frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}}\right)^2$$

Reject H_0 when this $\geq c$ as before

<u>Robustness</u>

Consider a location parameter θ $\qquad X_1, \dots, X_n \sim$ iid $f(x|\theta)$

$$f(x \mid \theta) = g(x-\theta)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i | \theta) = \prod_{i=1}^{n} g(x_i - \theta)$$

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$$\ell(\theta) = \ln L(\theta) = \sum_{i=1}^{n} \ln g(x_i - \theta)$$

Let $\rho(x_i - \theta) = -\ln g(x_i - \theta)$

So $\ell(\theta) = -\sum_{i=1}^{n} \rho(x_i - \theta)$

Then
$$\ell'(\theta) = -\sum_{i=1}^{n} \rho'(x_i - \theta)(-1)$$
$$= \sum_{i=1}^{n} \rho'(x_i - \theta)$$

Let $\psi(x_i - \theta) = \rho'(x_i - \theta)$

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So $\ell'(\theta) = \sum_{i=1}^{n} \psi(x_i - \theta)$

the MLE of θ is found by setting

$\sum_{i=1}^{n} \psi(x_i - \theta) = 0$ & solving for θ.

What if a different ψ function is used (other than $\psi = \rho'$).

Then we get an <u>M estimator</u>.

ψ is called the <u>influence</u> function.

(15)

Defn. If $\psi(x_i - \theta)$ is bounded, then the corresponding M estimator is <u>robust</u>.

Also $w(x-\theta) = \dfrac{\psi(x-\theta)}{x-\theta}$ is called the <u>weight function</u>

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(16)

<u>Normal example</u>

$$L(\theta) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$l(\theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$l'(\theta) = \cancel{-}\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(x_i-\theta)(\cancel{-1})$$

$$= \sum_{i=1}^{n}\frac{x_i-\theta}{\sigma^2}$$

So $\psi(x-\theta) = \dfrac{x-\theta}{\sigma^2}$ This is <u>not</u> bounded, so $\bar{x}$ is <u>not</u> <u>robust</u>.

$$\omega(x-\theta) = \frac{\psi(x-\theta)}{x-\theta} = \frac{1}{\sigma^2}$$ (17)