Some Asymptotic Results

Stat 563
5-21-26

Slutsky's Theorem

(1)

Suppose $X_n \xrightarrow{\mathcal{D}} X$

and $Y_n \xrightarrow{P} c$

Then (a) $X_n + Y_n \xrightarrow{\mathcal{D}} X + c$

(b) $X_n Y_n \xrightarrow{\mathcal{D}} cX$

Recall $X_n \xrightarrow{P} X$ if

$\forall \varepsilon > 0$

$$\lim_{n\to\infty} P[\,|X_n - X| \geq \varepsilon\,] = 0$$

$X_n \xrightarrow{\mathcal{D}} X$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

$\forall x$: x is a continuity point of $F(x)$

(2)

Proof: (a)

Assume $c = 0$

Let x be a point of continuity of $F(x)$

$$P(X_n + Y_n \leq x) = P(X_n + Y_n \leq x \cap |Y_n| \leq \varepsilon)$$

$$+ P(X_n + Y_n \leq x \cap |Y_n| > \varepsilon)$$

$$\leq P(X_n \leq x + \varepsilon) + P(|Y_n| > \varepsilon) \qquad (1)$$

Since $X_n \leq x - Y_n \cap -\varepsilon \leq Y_n \leq \varepsilon$

$\Rightarrow X_n \leq x - Y_n \cap \varepsilon \geq -Y_n \Rightarrow X_n \leq x + \varepsilon$

(3)

Also, $$P(X_n \le x-\varepsilon) = P(X_n \le x-\varepsilon \cap |Y_n| \le \varepsilon) + P(X_n \le x-\varepsilon \cap |Y_n| > \varepsilon)$$

$$\le P(X_n+Y_n \le x) + P(|Y_n| > \varepsilon) \quad (2)$$

Since

$$X_n \le x-\varepsilon \cap -\varepsilon \le Y_n \le \varepsilon \Rightarrow X_n+Y_n \le X_n+\varepsilon$$

$$\Rightarrow X_n+Y_n \le x$$

(4)

Recap:

(1) $P(X_n+Y_n \le x) \le P(X_n \le x+\varepsilon) + P(|Y_n| > \varepsilon)$

(2) $P(X_n \le x-\varepsilon) \le P(X_n+Y_n \le x) + P(|Y_n| > \varepsilon)$

(2): $P(X_n \le x-\varepsilon) - P(|Y_n| > \varepsilon) \le P(X_n+Y_n \le x)$

Then with (1): As $n \to \infty$

$$\underbrace{P(X_n \le x-\varepsilon)}_{\downarrow\, F(x-\varepsilon)} - \underbrace{P(|Y_n| > \varepsilon)}_{\downarrow\, 0} \le P(X_n+Y_n \le x) \le \underbrace{P(X_n \le x+\varepsilon)}_{\downarrow\, F(x+\varepsilon)} + \underbrace{P(|Y_n| > \varepsilon)}_{\downarrow\, 0}$$

so $\lim_{n\to\infty} P(X_n+Y_n \le x) = F(x)$

(5)

$\therefore X_n + Y_n \xrightarrow{D} X$

To finish (a), what if $c \neq 0$?

Write $X_n + Y_n = (X_n + c) + (Y_n - c)$

$= W_n + V_n$

Then $W_n \xrightarrow{D} X + c$ and $V_n \xrightarrow{P} 0$

$\Rightarrow W_n + V_n \xrightarrow{D} X + c$ This proves (a).

(6)

(b) First consider $c = 0$

$$P(|X_n Y_n| > \varepsilon) = P\left(|X_n Y_n| > \varepsilon \cap |Y_n| \leq \tfrac{1}{N}\right)$$

$$+ P\left(|X_n Y_n| > \varepsilon \cap |Y_n| > \tfrac{1}{N}\right)$$

$$\leq P(|X_n| > \varepsilon N) + P\left(|Y_n| > \tfrac{1}{N}\right)$$

Since $|X_n Y_n| > \varepsilon \cap |Y_n| \leq \frac{1}{N}$

$\Rightarrow |X_n| > \dfrac{\varepsilon}{|Y_n|} \cap N \leq \dfrac{1}{|Y_n|}$

$\Rightarrow |X_n| > \varepsilon N$

Take the limit as $n \to \infty$

$$P(|X| > \varepsilon N) \to 0$$

since $X_n \xrightarrow{D} X$

So $\displaystyle\lim_{n\to\infty} P(|X_nY_n| > \varepsilon) \le \underbrace{P(|X| > \varepsilon N)}$ $\forall \varepsilon, N$

This can be made as close to 0 as desired by N sufficiently large

$\therefore\ X_nY_n \xrightarrow{P} 0$

$\Rightarrow X_nY_n \xrightarrow{D} 0$

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To finish (b), what if $c \neq 0$?

Write $X_nY_n = X_n\underbrace{(Y_n - c)} + \underbrace{cX_n}$

$$X_n \xrightarrow{D} X, \qquad (Y_n - c) \xrightarrow{P} 0, \qquad cX_n \xrightarrow{D} cX$$

$$X_n(Y_n - c) \xrightarrow{P} 0$$

By Part (a), this converges in distribution to cX.

<u>Theorem</u>: <u>See the regularity conditions in Sec 10.6</u>

Let $\hat{\theta}$ be the MLE of θ.

Then $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N\left(0, \frac{1}{I(\theta)}\right)$

That is, MLEs are <u>asymptotically</u> unbiased, efficient, and normally distributed.

<u>Proof</u>: 1st order Taylor series with remainder term:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x^*)\frac{(x-x_0)^2}{2}$$

where x^* lies between x and x_0

Do this for $l'(\hat{\theta})$, expanded around θ

$l(\theta)$ is the log-likelihood function

$$\underset{0}{\underbrace{l'(\hat{\theta})}} = l'(\theta) + l''(\theta)(\hat{\theta} - \theta) + l'''(\theta^*)\frac{(\hat{\theta} - \theta)^2}{2}$$

where θ^* lies between $\hat{\theta}$ and θ

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$$-\ell'(\theta) = (\hat{\theta}-\theta)\left[\ell''(\theta) + \ell'''(\theta^*)\,\frac{\hat{\theta}-\theta}{2}\right]$$

$$\hat{\theta}-\theta = \frac{-\ell'(\theta)}{\ell''(\theta)+\ell'''(\theta^*)\,\frac{\hat{\theta}-\theta}{2}}$$

$$\sqrt{n}\,(\hat{\theta}-\theta) = \frac{-\ell'(\theta)}{\ell''(\theta)+\ell'''(\theta^*)\,\frac{\hat{\theta}-\theta}{2}} \quad \cancel{\sqrt{n}} \quad \cancel{\frac{\sqrt{n}}{\sqrt{n}}} \quad \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}}$$

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$$\sqrt{n}\,(\hat{\theta}-\theta) = \frac{-\frac{1}{\sqrt{n}}\,\ell'(\theta)}{\frac{1}{n}\ell''(\theta)+\frac{1}{n}\ell'''(\theta^*)\,\frac{\hat{\theta}-\theta}{2}}$$