Stat 563
5-12-26

(1)

$$V[\hat{\beta}_1] = \sigma^2 \sum_{i=1}^{n} c_i^2 \quad , \quad \sum c_i x_i = 1, \quad \sum c_i = 0$$

& we wanted to minimize $\sum c_i^2$

The Lemma said

$$\frac{\left(\sum a_i v_i\right)^2}{\sum a_i^2/w_i} \text{ is maximized when}$$

$$a_i = k\, w_i (v_i - \bar{v}_w)$$

$$\text{where } \bar{v}_w = \frac{\sum w_i v_i}{\sum w_i}$$

(2)

In the lemma, let $v_i = x_i$, $a_i = c_i$, $w_i \equiv 1$

Then

$$\frac{\left(\sum c_i x_i\right)^2}{\sum c_i^2} \text{ is maximized when}$$

$$c_i = k(x_i - \bar{x})$$

But $\sum c_i x_i = 1$

So $\sum c_i^2$ is minimized when $c_i = k(x_i - \bar{x})$

$$\sum c_i x_i = 1 \Rightarrow \sum k(x_i - \bar{x}) x_i = 1$$

③

$$k \underbrace{\sum x_i(x_i-\bar{x})}_{S_{xx}} = 1$$

So $k = \dfrac{1}{S_{xx}}$

$$\Rightarrow \quad c_i = \frac{1}{S_{xx}}(x_i - \bar{x})$$

And $\hat{\beta}_1 = \sum c_i Y_i = \sum \frac{1}{S_{xx}}(x_i - \bar{x}) Y_i$

$$= \frac{1}{S_{xx}} \underbrace{\sum (x_i - \bar{x}) Y_i}_{S_{xY}} = \frac{S_{xY}}{S_{xx}}$$

$\therefore$ $\hat{\beta}_1$ is the BLUE of β_1

④

We know $E[\hat{\beta}_1] = \beta_1$

$$V[\hat{\beta}_1] = \sigma^2 \sum c_i^2 \qquad c_i = k(x_i - \bar{x}), \quad k = \frac{1}{S_{xx}}$$

$$= \sigma^2 \frac{1}{S_{xx}^2} \underbrace{\sum (x_i - \bar{x})^2}_{S_{xx}}$$

$$= \frac{\sigma^2}{S_{xx}}$$

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$. This is also a linear estimator.

$$= \frac{1}{n}\sum Y_i - \frac{S_{xY}}{S_{xx}} \bar{x}$$

⑤

$$= \frac{1}{n}\sum Y_i - \frac{1}{S_{xx}}\bar{x}\sum(x_i-\bar{x})Y_i$$

$$= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{S_{xx}}\right)Y_i \quad \therefore \underline{\text{linear}}$$

$$E[\hat{\beta}_0] = E[\bar{Y} - \hat{\beta}_1\bar{x}] = E[\bar{Y}] - \bar{x}E[\hat{\beta}_1]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(\beta_0+\beta_1 x_i+\varepsilon_i)\right] - \bar{x}\beta_1$$

$$= \beta_0 + \cancel{\beta_1\bar{x}} + 0 - \cancel{\beta_1\bar{x}} = \beta_0$$

⑥

$$V[\hat{\beta}_0] = V\left[\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{S_{xx}}\right)Y_i\right]$$

$$= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{S_{xx}}\right)^2 \underbrace{V[Y_i]}_{\sigma^2}$$

$$= \sigma^2\sum_{i=1}^{n}\left(\frac{1}{n^2} + \frac{\bar{x}^2(x_i-\bar{x})^2}{S_{xx}^2} - \frac{2}{n}\frac{\bar{x}(x_i-\bar{x})}{S_{xx}}\right)$$

$$= \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}^2}\underbrace{\sum(x_i-\bar{x})^2}_{S_{xx}} - \frac{2\bar{x}}{nS_{xx}}\underbrace{\sum(x_i-\bar{x})}_{0}\right]$$

$$= \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]$$

⑦

Since $\hat{\beta}_1$ and $\hat{\beta}_0$ are both linear combinations of the Y_i's and the Y_i's were indep normal random variables, they are both normally distributed.

$$\therefore \quad \hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{S_{xx}}\right)$$

and $$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$$

⑧

Our MLE of σ^2 was

$$\hat{\sigma}^2 = \frac{S_{yy} - \frac{S_{xy}^2}{S_{xx}}}{n}$$

Find $E[\hat{\sigma}^2]$

$$S_{yy} = \sum (Y_i - \bar{Y})^2$$
$$= \sum \bar{Y}(Y_i - \bar{Y})$$
$$= \sum Y_i^2 - \frac{(\sum Y_i)^2}{n}$$

$$E[S_{yy}] = E\left[\sum Y_i^2 - n\bar{Y}^2\right]$$

$$E[Y_i^2] = V[Y_i] + (E[Y_i])^2$$
$$= \sigma^2 + (\beta_0 + \beta_1 x_i)^2$$

$$E[\bar{Y}^2] = V[\bar{Y}] + (E[\bar{Y}])^2 = \frac{\sigma^2}{n} + (\beta_0 + \beta_1 \bar{x})^2$$

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$$\text{Now } E[S_{yy}] = \sum\left(\sigma^2 + (\beta_0+\beta_1 x_i)^2\right)$$
$$- n\left[\frac{\sigma^2}{n} + (\beta_0+\beta_1\bar{x})^2\right]$$
$$= n\sigma^2 + n\beta_0^2 + \beta_1^2\sum x_i^2 + \cancel{2\beta_0\beta_1\sum x_i}$$
$$- \sigma^2 - n\beta_0^2 - n\beta_1^2\bar{x}^2 - \cancel{2n\beta_0\beta_1\bar{x}}$$
$$= (n-1)\sigma^2 + \beta_1^2\left[\underbrace{\sum x_i^2 - n\bar{x}^2}_{S_{xx}}\right]$$
$$E[S_{yy}] = (n-1)\sigma^2 + \beta_1^2 S_{xx}$$

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$$E\left[\frac{S_{xy}^2}{S_{xx}}\right] = E\left[\frac{S_{xy}^2}{S_{xx}^2} S_{xx}\right]$$
$$= S_{xx} E\left[\hat{\beta}_1^2\right]$$
$$= S_{xx}\left[V[\hat{\beta}_1] + \left(E[\hat{\beta}_1]\right)^2\right]$$
$$= S_{xx}\left[\frac{\sigma^2}{S_{xx}} + \beta_1^2\right]$$
$$E[\hat{\sigma}^2] = E\left[\frac{S_{yy} - \frac{S_{xy}^2}{S_{xx}}}{n}\right]$$
$$= \frac{1}{n}\left[(n-1)\sigma^2 + \cancel{\beta_1^2 S_{xx}} - \sigma^2 - \cancel{\beta_1^2 S_{xx}}\right] = \frac{n-2}{n}\sigma^2$$

(11)

Define $s^2 = \frac{n}{n-2}\hat{\sigma}^2 = \dfrac{S_{YY} - \frac{S_{XY}^2}{S_{XX}}}{n-2}$

Then $E[s^2] = \sigma^2$