Stat 563
5-7-26

Regression MLEs, Continued

(1)

We had
$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{S_{xx}}$$

$$= \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{S_{xx}}$$

$$= \frac{\sum_i (x_i - \bar{x}) y_i - \bar{y} \overbrace{\sum_i (x_i - \bar{x})}^{\sum x_i - n\bar{x} = 0}}{S_{xx}}$$

$$\hat{\beta}_1 = \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i - \bar{x}) y_i = \sum_{i=1}^{n} \underbrace{\frac{x_i - \bar{x}}{S_{xx}}}_{c_i} y_i = \sum_{i=1}^{n} c_i y_i$$

(2)

Defn: A linear estimator is an estimator that is a linear combination of the observations.

Defn: A BLUE (Best linear unbiased estimator) is of the form $\sum_{i=1}^{n} c_i y_i$, is unbiased, and its variance is less than or equal to the variance of any other linear unbiased estimator.

③

For $\hat{\beta}_1$ to be unbiased, we need

$$E[\hat{\beta}_1] = \beta_1$$

$$\shortparallel$$

$$E[\sum_{i=1}^{n} c_i y_i] = \sum_{i=1}^{n} c_i E[Y_i]$$

$$= \sum_{i=1}^{n} c_i E[\beta_0 + \beta_1 x_i + \varepsilon_i]$$

Need:

$$\beta_1 = \sum_{i=1}^{n} c_i(\beta_0 + \beta_1 x_i) = \beta_0 \sum_{i=1}^{n} c_i + \beta_1 \sum_{i=1}^{n} c_i x_i \quad \forall \beta_0, \beta_1$$

So $\sum c_i x_i = 1$ and $\sum_{i=1}^{n} c_i = 0$

④

$$\text{Also, } V[\hat{\beta}_1] = V[\sum_{i=1}^{n} c_i Y_i]$$

$$= \sum_{i=1}^{n} c_i^2 V[Y_i]$$

$$= \sum_{i=1}^{n} c_i^2 V[\beta_0 + \beta_1 X_i + \varepsilon_i]$$

$$= \sigma^2 \sum_{i=1}^{n} c_i^2$$

$\therefore$ The BLUE of β_1 must have $\sum c_i^2$ minimized,

subject to $\sum c_i x_i = 1$ and $\sum c_i = 0$.

⑤

Lemma: Let $v_1, \dots, v_n$ be constants.

Let $w_1, \dots, w_n$ be positive constants.

Let $A = \left\{ \vec{a} = (a_1, \dots, a_n) : \sum_{i=1}^{n} a_i = 0 \right\}$

Then $$\max_A \left\{ \frac{\left(\sum_{i=1}^{n} a_i v_i\right)^2}{\sum a_i^2/w_i} \right\} = \sum_{i=1}^{n} w_i (v_i - \bar{v}_w)^2,$$

where $$\bar{v}_w = \frac{\sum_{i=1}^{n} w_i v_i}{\sum_{i=1}^{n} w_i}.$$

Also, the maximum is achieved for any $\vec{a}$ of the form $(a_1, \dots, a_n)$ where $a_i = k w_i (v_i - \bar{v}_w)$, $k \neq 0$

⑥

Proof: Let $$B = \left\{ \vec{b} = (b_1, \dots, b_n) : \sum_{i=1}^{n} b_i = 0 \text{ and } \sum_{i=1}^{n} \frac{b_i^2}{w_i} = 1 \right\}$$

Choose an $\vec{a} \in A$.

Define $$b_i = \frac{a_i}{\sqrt{\sum_{i=1}^{n} \frac{a_i^2}{w_i}}}$$

Then $$\sum b_i = \frac{1}{\sqrt{\sum \frac{a_i^2}{w_i}}} \sum a_i = 0$$

and $$\sum \frac{b_i^2}{w_i} = \frac{1}{\sum \frac{a_i^2}{w_i}} \sum \frac{a_i^2}{w_i} = 1, \text{ so } \vec{b} \in B$$

⑦

Also
$$\frac{(\sum a_i v_i)^2}{\sum \frac{a_i^2}{w_i}} = \left(\frac{\sum a_i v_i}{\sqrt{\sum \frac{a_i^2}{w_i}}}\right)^2 = \left(\sum b_i v_i\right)^2$$

Let $W = \sum w_i$

Then
$$\frac{1}{W^2}\left(\sum b_i v_i\right)^2 = \left(\sum \frac{b_i}{w_i} v_i \frac{w_i}{W}\right)^2$$

Define random variables X and V so that
$$P\left(X = \frac{b_i}{w_i} \cap V = v_i\right) = \frac{w_i}{W}, \quad i = 1, \ldots, n$$

⑧

$$E[XV] = \sum \frac{b_i}{w_i} v_i \cdot \frac{w_i}{W}$$

and
$$E[X] = \sum \frac{b_i}{w_i} \cdot \frac{w_i}{W} = \frac{1}{W} \sum b_i = 0$$

$$E[V] = \sum v_i \frac{w_i}{W} = \frac{\sum v_i w_i}{\sum w_i} = \bar{v}_w$$

Now
$$\begin{aligned}\frac{1}{W^2}\left(\sum b_i v_i\right)^2 &= \left(E[XV]\right)^2 \\ &= \left[\mathrm{Cov}(X,V)\right]^2 \\ &\le V[X]\, V[V] \quad (\text{since } \rho^2 \le 1)\end{aligned}$$

(9)

$$\frac{1}{W^2}\left(\sum b_i v_i\right)^2 \le \underbrace{\sum \frac{b_i^2}{w_i^2}\frac{w_i}{W}}_{E[X^2]=V[X]} \cdot \underbrace{\sum (v_i - \bar{v}_w)^2 \cdot \frac{w_i}{W}}_{Var[V]}$$

$$\left(\sum b_i v_i\right)^2 \le \underbrace{\left(\sum \frac{b_i^2}{w_i}\right)}_{1}\left(\sum (v_i - \bar{v}_w)^2 w_i\right)$$

$$\left(\sum b_i v_i\right)^2 = \frac{\left(\sum a_i v_i\right)^2}{\sum \frac{a_i^2}{w_i}}$$

This gives us our upper bound. We only need to show equality when $a_i = k w_i (v_i - \bar{v}_w)$

(10)

$$\sum a_i = k \sum w_i (v_i - \bar{v}_w)$$
$$= k\left(\sum w_i v_i - \underbrace{\bar{v}_w}_{\frac{\sum w_i v_i}{\sum w_i}} \sum w_i\right) = 0$$

So $\vec{a} \in A$

$$b_i = \frac{\cancel{k} w_i (v_i - \bar{v}_w)}{\sqrt{\sum \frac{(\cancel{k} w_i (v_i - \bar{v}_w))^2}{w_i}}} = \frac{w_i (v_i - \bar{v}_w)}{\sqrt{\sum w_i (v_i - \bar{v}_w)^2}}$$

$$\left(\sum b_i v_i\right)^2 = \left(\sum \frac{w_i (v_i - \bar{v}_w) v_i}{\sqrt{\sum w_i (v_i - \bar{v}_w)^2}}\right)^2$$

$$= \frac{1}{\sum w_i (v_i - \bar{v}_w)^2} \Big(\underbrace{\sum w_i (v_i - \bar{v}_w) v_i}_{\star} \Big)^2$$

We had $\sum w_i (v_i - \bar{v}_w) = 0$

so $\sum w_i (v_i - \bar{v}_w)(v_i - \bar{v}_w)$

$$= \star - \bar{v}_w \underbrace{\sum w_i (v_i - \bar{v}_w)}_{0}$$

$= \sum w_i (v_i - \bar{v}_w)^2$, which establishes the equality. ✓