Sequential probability ratio test, continued. Stat 563 5-5-26

Remaining question was: How are k_0 & k_1 chosen? ①

Let $R_n = \{\vec{x} : k_0 < \Lambda_j < k_1 \ \forall \ 1 \le j \le n-1$ and $\Lambda_n \le k_0\}$

$A_n = \{\vec{x} : k_0 < \Lambda_j < k_1 \ \forall \ 1 \le j \le n-1$ and $\Lambda_n \ge k_1\}$

$$\alpha = \text{Prob (Type I error)}$$
$$= \text{Prob}(\text{Rej } H_0 \mid H_0) = P\left(\bigcup_{n=1}^{\infty} R_n \mid H_0\right)$$

②

$$\alpha = \sum_{n=1}^{\infty} P(R_n \mid H_0) = \sum_{n=1}^{\infty} \int_{R_n} L(\theta_0)\, d\vec{x}$$

Note: for $\vec{x} \in R_n$, $\Lambda_n \le k_0$

$$\Lambda_n = \frac{L(\theta_0)}{L(\theta_1)}$$

So, for $\vec{x} \in R_n$, $L(\theta_0) \le k_0 L(\theta_1)$

$$\alpha \le \sum_{n=1}^{\infty} \int_{R_n} k_0 L(\theta_1)\, d\vec{x} = \sum_{n=1}^{\infty} k_0 P(\vec{x} \in R_n \mid H_1)$$

③

$$\alpha \leq k_0 \sum_{n=1}^{\infty} P(\vec{x} \in R_n \mid H_1)$$

Let t_n = Prob(procedure terminates at step $n \mid H_1$)

$$= P(R_n \mid H_1) + P(A_n \mid H_1)$$

$$\alpha \leq k_0 \sum_{n=1}^{\infty} \left(t_n - P(A_n \mid H_1)\right)$$

$$= k_0 \left(\sum_{n=1}^{\infty} t_n - \sum_{n=1}^{\infty} P(A_n \mid H_1) \right)$$

$$\leq k_0 \left(1 - P\left(\bigcup_{n=1}^{\infty} A_n \mid H_1 \right) \right)$$ ④

$$\alpha \leq k_0 (1 - \beta)$$

Also, $1 - \alpha = P(\text{Acc } H_0 \mid H_0)$

$$= P\left(\bigcup_{n=1}^{\infty} A_n \mid H_0 \right)$$

$$= \sum_{n=1}^{\infty} P(A_n \mid H_0)$$

$$= \sum_{n=1}^{\infty} \int_{A_n} L(\theta_0)\, d\vec{x}$$

(5)

Note: For $\vec{x} \in A_n$, $\Lambda_n \geq k_1$

$$\Lambda_n = \frac{L(\theta_0)}{L(\theta_1)}$$

$$L(\theta_0) \geq k_1 L(\theta_1)$$

$$1-\alpha \geq \sum_{n=1}^{\infty} \int_{A_n} k_1 L(\theta_1)\, d\vec{x}$$

$$= k_1 \sum_{n=1}^{\infty} P(\vec{x} \in A_n \mid H_1)$$

$$= k_1 P\left(\bigcup_{n=1}^{\infty} A_n \mid H_1\right) = k_1 \beta$$

(6)

So $\alpha \leq k_0(1-\beta)$ and $1-\alpha \geq k_1\beta$

$$\frac{\alpha}{1-\beta} \leq k_0 \quad \text{and} \quad \frac{1-\alpha}{\beta} \geq k_1$$

Choose a desired α and β, called α^* and β^*

Let $k_0 = \dfrac{\alpha^*}{1-\beta^*}$ and $k_1 = \dfrac{1-\alpha^*}{\beta^*}$

Then $\dfrac{\alpha}{1-\beta} \leq \dfrac{\alpha^*}{1-\beta^*}$ and $\dfrac{1-\alpha}{\beta} \geq \dfrac{1-\alpha^*}{\beta^*}$

Does this imply that $\alpha \leq \alpha^*$ and $\beta \leq \beta^*$?

⑦

$$\alpha(1-\beta^*) \le \alpha^*(1-\beta) \text{ and } (1-\alpha)\beta^* \ge \beta(1-\alpha^*)$$

$$\alpha - \alpha\beta^* \le \alpha^* - \alpha^*\beta$$
$$\beta - \alpha^*\beta \le \beta^* - \alpha\beta^*$$

$$\alpha + \beta - \cancel{\alpha\beta^*} - \cancel{\alpha^*\beta} \le \alpha^* + \beta^* - \cancel{\alpha^*\beta} - \cancel{\alpha\beta^*}$$

So $\underline{\alpha + \beta \le \alpha^* + \beta^*}$

⑧

<u>Look at simple linear regression as maximum likelihood problem.</u>

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \qquad \varepsilon_i \sim \text{iid } N(0,\sigma^2)$$
$$i = 1,\dots,n$$

X_i: known constants

$Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$, all independent

$$L(\beta_0,\beta_1,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i-\beta_0-\beta_1x_i)^2}$$

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2} \qquad (9)$$

$$\ell(\beta_0,\beta_1,\sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2$$

$$\frac{\partial \ell}{\partial \beta_0} = \cancel{-}\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(y_i-\beta_0-\beta_1 x_i)(\cancel{-1}) \overset{\text{set}}{=} 0$$

$$\sum y_i - n\beta_0 - \beta_1\sum x_i = 0$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

$$(10)$$

$$\frac{\partial \ell}{\partial \beta_1} = \cancel{-}\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(y_i-\beta_0-\beta_1 x_i)(\cancel{-}x_i) \overset{\text{set}}{=} 0$$

$$\sum x_i y_i - \beta_0\sum x_i - \beta_1\sum x_i^2 = 0$$

$$\sum x_i y_i - (\bar{y}-\beta_1\bar{x})\sum x_i - \beta_1\sum x_i^2 = 0$$

$$\sum x_i y_i - \bar{y}\sum x_i = \beta_1\left(\sum x_i^2 - \bar{x}\sum x_i\right)$$

$$\hat{\beta}_1 = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2} = \frac{S_{xy}}{S_{xx}}$$

(11)

Note: These are the same $\hat{\beta}_0$ and $\hat{\beta}_1$ that minimize the error sum of squares.

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{\cancel{2}} \frac{\cancel{2\pi}}{\cancel{2\pi}\sigma^2} - \frac{1}{\cancel{2}} \frac{-1}{\sigma^4} \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \overset{\text{set}}{=} 0$$

$$-n\sigma^2 + \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = 0$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - (\bar{y} - \hat{\beta}_1 \bar{x}) - \hat{\beta}_1 x_i)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \bar{y} - \hat{\beta}_1 (x_i - \bar{x}) \right)^2$$

(12)

$$\hat{\sigma}^2 = \frac{1}{n}\Big[\underbrace{\sum_{i=1}^{n} (y_i - \bar{y})^2}_{S_{yy}} + \hat{\beta}_1^2 \overbrace{\sum_{i=1}^{n} (x_i - \bar{x})^2}^{S_{xx}} - 2\hat{\beta}_1 \underbrace{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}_{S_{xy}} \Big]$$

$$\hat{\sigma}^2 = \frac{1}{n}\left[S_{yy} + \hat{\beta}_1^2 S_{xx} - 2\hat{\beta}_1 S_{xy} \right]$$

$$= \frac{1}{n}\left[S_{yy} + \left(\frac{S_{xy}}{S_{xx}}\right)^2 S_{xx} - 2\left(\frac{S_{xy}}{S_{xx}}\right) S_{xy} \right]$$

$$\hat{\sigma}^2 = \frac{1}{n}\left[S_{yy} - \frac{S_{xy}^2}{S_{xx}} \right]$$

(This disagrees with $MSE = \frac{SSE}{n-2}$)