Stat 5263
4-28-26

Continuation of example from last time.

①

$H_0: \mu = 6$ $\quad \bar{x} = 6.7 \quad n = 18 \quad \sigma = 2$

$H_1: \mu > 6$ $\quad$ Test stat $= \dfrac{6.7-6}{2/\sqrt{18}} = 1.48$

$\alpha = .05$

∴ we failed to reject H_0 $\quad$ pval $= .069$

Now, find the power of the test if $\mu = 7.5$.

$\delta = \mu_1 - \mu_0$

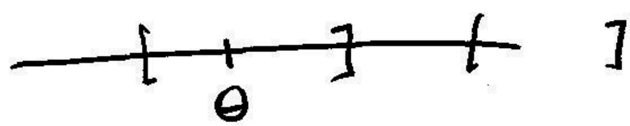
$$\text{Power} = P\left(Z > z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right)$$
$$= P\left(Z > 1.645 - \frac{1.5\sqrt{18}}{2}\right)$$
$$= P(Z > -1.53) = .937$$

②

Defn: An *interval estimate* of θ is

$$[L(\vec{X}), U(\vec{X})]$$

$X_1, \dots, X_n \sim$ iid $f(x|\theta)$



Defn: The *coverage probability* is

$$P_\theta[L(\vec{X}) \le \theta \le U(\vec{X})]$$

Defn: The *confidence* associated with an interval estimate is $\inf_{\theta \in \Omega} P_\theta[L(\vec{X}) \le \theta \le U(\vec{X})]$

③

<u>Theorem</u>: $\forall\, \theta_0 \in \Omega$, let $A(\theta_0)$ be the acceptance region (the complement of the rejection region) of a level α test of $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$.

$\forall\, \vec{x}$, define $C(\vec{x}) = \{\theta_0 : \vec{x} \in A(\theta_0)\}$

Then $C(\vec{X})$ is a $1-\alpha$ confidence region.

④

<u>Pf</u>: $P_{\theta_0}(\vec{X} \notin A(\theta_0)) \le \alpha$

So $P_{\theta_0}(\vec{X} \in A(\theta_0)) \ge 1-\alpha$

then $P_{\theta_0}[\theta_0 \in C(\vec{X})] = P_{\theta_0}[\vec{X} \in A(\theta_0)]$

$$\ge 1-\alpha$$

This was true $\forall\, \theta_0$

$\therefore$ $C(\vec{X})$ has confidence at least $1-\alpha$

(5)

<u>Example</u>: $X_1, \dots, X_n \sim \text{iid } N(\mu, \underset{\uparrow \text{known}}{\sigma^2})$

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

From a previous example, our decision rule said to Reject H_0 if $\left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| > Z_{\alpha/2}$

The acceptance region is

$$\left\{\vec{x} : \left|\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right| \leq Z_{\alpha/2}\right\}$$

(6)

$$= \left\{\vec{x} : -Z_{\alpha/2} \leq \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \leq Z_{\alpha/2}\right\}$$

$$= \left\{\vec{x} : \bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{x} + Z_{\alpha/2}\sigma/\sqrt{n}\right\}$$

A $1-\alpha$ confidence region for μ is

$$\Big[\underbrace{\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{L(\vec{X})}, \underbrace{\bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{U(\vec{X})}\Big]$$

(7)

Ex: $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ ← known

$H_0: \mu \le \mu_0$

$H_1: \mu > \mu_0$ The rejection region is

$$\left\{ \vec{x} : \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha \right\}$$

The acceptance region is $\left\{ \vec{x} : \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \le z_\alpha \right\}$

$$= \left\{ \vec{x} : \mu_0 \ge \bar{x} - z_\alpha \sigma/\sqrt{n} \right\}$$

So $\left[\bar{X} - z_\alpha \sigma/\sqrt{n}, \infty \right)$ is a $1-\alpha$ conf. region

(8)

Sequential Probability Ratio Test

$H_0: \theta = \theta_0$

$H_1: \theta = \theta_1$

Let k_0 and k_1 be 2 constants such that $k_0 < k_1$

Observe X_1

Compute $\Lambda_1 = \dfrac{L(\theta_0)}{L(\theta_1)}$ based on just 1 obs.

Rule: If $\Lambda_1 \le k_0$ reject H_0

If $\Lambda_1 \ge k_1$ fail to reject H_0

If $k_0 < \Lambda_1 < k_1$, collect another observation

Repeat until H_0 is rejected or accepted (9)

Λ_n is based on $X_1, \ldots, X_n$

Example: $f(x|\theta) = \theta^x (1-\theta)^{1-x} \quad x = 0,1$

(Bernoulli)

$H_0: \theta = \frac{1}{3}$

$H_1: \theta = \frac{2}{3}$

$$L(\theta) = \theta^{\Sigma x_i} (1-\theta)^{n - \Sigma x_i}$$

$$\Lambda_1 = \frac{\frac{1}{3}^{x_1} \; \frac{2}{3}^{1-x_1}}{\frac{2}{3}^{x_1} \; \frac{1}{3}^{1-x_1}} = 2^{1-2x_1}$$

(10)

$$\Lambda_n = \frac{\frac{1}{3}^{\Sigma x_i} \; \frac{2}{3}^{n-\Sigma x_i}}{\frac{2}{3}^{\Sigma x_i} \; \frac{1}{3}^{n-\Sigma x_i}} = 2^{n-2\Sigma x_i}$$

Rule at each step:

Reject H_0 if $\quad 2^{n-2\Sigma x_i} \le k_0$

Accept H_0 if $\quad 2^{n-2\Sigma x_i} \ge k_1$

Keep going if $\quad k_0 < 2^{n-2\Sigma x_i} < k_1$

Equivalently,

Reject H_0 if $\sum x_i \geq k_0'$

Accept H_0 if $\sum x_i \leq k_1'$

Keep going if $k_1' < \sum x_i < k_0'$

Remaining problem:

How do we determine k_0' and k_1' ?

Stat 563 HW #4

9.4 Let $X_1, \dots, X_n$ be a random sample from a $\mathrm{n}(0, \sigma_X^2)$, and let $Y_1, \dots, Y_m$ be a random sample from a $\mathrm{n}(0, \sigma_Y^2)$, independent of the Xs. Define $\lambda = \sigma_Y^2/\sigma_X^2$.

(a) Find the level α LRT of $H_0\colon \lambda = \lambda_0$ versus $H_1\colon \lambda \neq \lambda_0$.

(b) Express the rejection region of the LRT of part (a) in terms of an F random variable.

(c) Find a $1-\alpha$ confidence interval for λ.

9.7 (a) Find the $1-\alpha$ confidence set for a that is obtained by inverting the LRT of $H_0: a = a_0$ versus $H_1: a \neq a_0$ based on a sample $X_1, \dots, X_n$ from a $\mathrm{n}(\theta, a\theta)$ family, where θ is unknown.

(b) A similar question can be asked about the related family, the $\mathrm{n}(\theta, a\theta^2)$ family. If $X_1, \dots, X_n$ are iid $\mathrm{n}(\theta, a\theta^2)$, where θ is unknown, find the $1-\alpha$ confidence set based on inverting the LRT of $H_0\colon a = a_0$ versus $H_1\colon a \neq a_0$.

9.17 Find a $1-\alpha$ confidence interval for θ, given $X_1, \dots, X_n$ iid with pdf

(a) $f(x|\theta) = 1,\ \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$.

(b) $f(x|\theta) = 2x/\theta^2,\ 0 < x < \theta,\ \theta > 0$.