Stat 5263
4-23-26

(1)

Additional comment on the Bayesian hypothesis test from last time.

Instead of rejecting H_0 when $P(H_1) \geq \frac{1}{2}$,

Consider this:

Suppose the cost of a Type I error is l_{01}

" " " " " " II " " l_{10}

These are the <u>losses</u>

Recall that the <u>risk</u> is the expected loss $E[\mathcal{L}]$

$$Risk = l_{01} P(\text{Type I}) + l_{10} P(\text{Type II})$$

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$$Risk = l_{01} P(H_0 \text{ is true} \cap \text{Reject } H_0) + l_{10} P(H_1 \text{ is true} \cap \text{Fail to reject } H_0)$$

$$= l_{01} P(H_0 \text{ is true}) P(\text{Reject } H_0 \mid H_0 \text{ is true}) + l_{10} P(H_1 \text{ is true}) P(\text{Fail to reject } H_0 \mid H_1 \text{ is true})$$

Then the Bayesian rule will be adjusted to minimize this risk.

Let $p_0 = P(H_0)$ (prior probability of H_0)

$p_1 = P(H_1)$ $(p_0 + p_1 = 1)$

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Conditional risk, given H_0 is

$$R_0 = E[\mathcal{L} \mid H_0]$$
$$= 0(1-\alpha) + l_{01}\alpha + l_{10}\cdot 0$$
$$= l_{01}\alpha$$

Conditional risk, given H_1 is

$$R_1 = E[\mathcal{L} \mid H_1]$$
$$= 0(1-\beta) + l_{01}\cdot 0 + l_{10}\beta = l_{10}\beta$$

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Define $ECM = p_0 R_0 + p_1 R_1$

"expected cost of misclassification"

We would like to minimize this total risk

$$ECM = p_0 l_{01}\alpha + p_1 l_{10}\beta$$
$$= p_0 l_{01}\int_R L(\theta_0)\,d\vec{x} + p_1 l_{10}\int_A L(\theta_1)\,d\vec{x}$$

$$ECM = p_0 l_{01}\left[1 - \int_A L(\theta_0)\,d\vec{x}\right] + p_1 l_{10}\int_A L(\theta_1)\,d\vec{x}$$

$$= p_0 l_{01} + \int_A p_0 l_{01}(-L(\theta_0)) + p_1 l_{10} L(\theta_1)\quad d\vec{x}$$

$$= p_0 l_{01} + \int_A \left(p_1 l_{10} L(\theta_1) - p_0 l_{01} L(\theta_0)\right)\,d\vec{x}$$

We can minimize ECM by choosing A

s.t. that the integral is minimized

That is, define

$$A = \left\{\vec{x} : p_1 l_{10} L(\theta_1) - p_0 l_{01} L(\theta_0) < 0\right\}$$

$$= \left\{\vec{x} : p_1 l_{10} L(\theta_1) < p_0 l_{01} L(\theta_0)\right\}$$

$$= \left\{\vec{x} : \frac{L(\theta_0)}{L(\theta_1)} > \frac{p_1 l_{10}}{p_0 l_{01}}\right\}$$

$$= \left\{\vec{x} : \Lambda > \frac{p_1 l_{10}}{p_0 l_{01}}\right\}$$

So reject H_0 when $\Lambda \leq \dfrac{p_1 l_{10}}{p_0 l_{01}}$

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Power & p-values

Recall: $\alpha = \text{Prob}(\text{Rej } H_0 \mid H_0 \text{ true})$

$\beta = \text{Prob}(\text{Fail to rej. } H_0 \mid H_1 \text{ is true})$

The power of the test is $1-\beta =$

$\text{Prob}(\text{rej } H_0 \mid H_1 \text{ is true})$

Example: $H_0: \mu = \mu_0$ $\quad X_1, \dots, X_n \sim \text{iid } N(\mu, \sigma^2)$

$H_1: \mu > \mu_0$ $\quad$ (σ^2 known)

L.R.T. said to reject H_0 when $\bar{x} > c$

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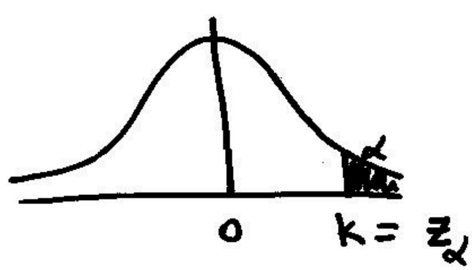
Given that H_0 is true,

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\bar{X} > c$ is equivalent to $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > k$



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Our rejection rule is $\dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > Z_\alpha$

Select a specific $\mu_1 > \mu_0$

$$\begin{aligned}
\text{Power} &= P(\text{Reject } H_0 \mid \mu=\mu_1) \\
&= P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > Z_\alpha \,\middle|\, \mu=\mu_1\right) \\
&= P\left(\bar{X} > \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \,\middle|\, \mu=\mu_1\right) \\
&= P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} > \frac{\mu_0-\mu_1+Z_\alpha \sigma/\sqrt{n}}{\sigma/\sqrt{n}} \,\middle|\, \mu=\mu_1\right)
\end{aligned}$$

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$$= P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} > Z_\alpha - \frac{(\mu_1-\mu_0)\sqrt{n}}{\sigma} \,\middle|\, \mu=\mu_1\right)$$

$$\text{Power} = P\left(Z > Z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right) \quad \text{where } \delta = \mu_1-\mu_0$$

Note: as $n \uparrow$ Power $\uparrow$

as $\mu_1 \uparrow$ Power $\uparrow$

as $\alpha \downarrow$ Power $\downarrow$

If σ is large, power will be small

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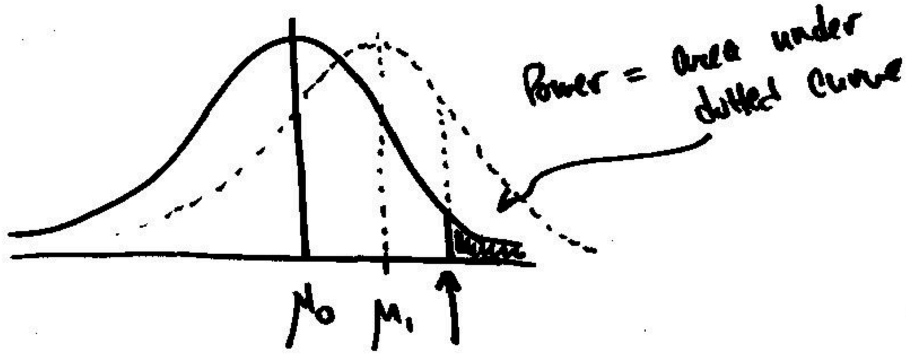

Defn: A test whose power never drops below α is called unbiased

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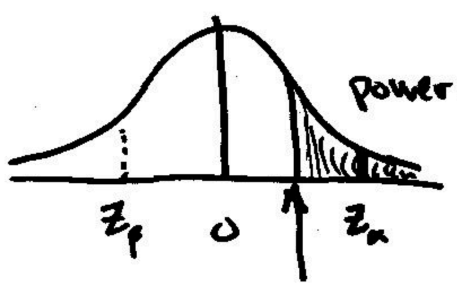
Problem: For a specific δ, find the sample size necessary to achieve a given power.

$$\text{Power} = P\left(Z > z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right)$$



$$z_\alpha - \frac{\delta\sqrt{n}}{\sigma} = z_{1-\beta} = -z_\beta$$

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So set $Z_\alpha - \frac{\delta\sqrt{n}}{\sigma} = -Z_\beta$ + solve for n

$$Z_\alpha + Z_\beta = \frac{\delta\sqrt{n}}{\sigma}$$

$$n = \left[\frac{(Z_\alpha + Z_\beta)\sigma}{\delta}\right]^2$$

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Defn: The p-value is the probability that the test statistic $W(\vec{X})$ is more extreme than the observed value $w(\vec{x})$, given H_0.

Example, continued $H_0: \mu = 6$ $\alpha = .05$
$H_1: \mu > 6$

Suppose we observe $\bar{x} = 6.7$, $n = 18$, $\sigma = 2$

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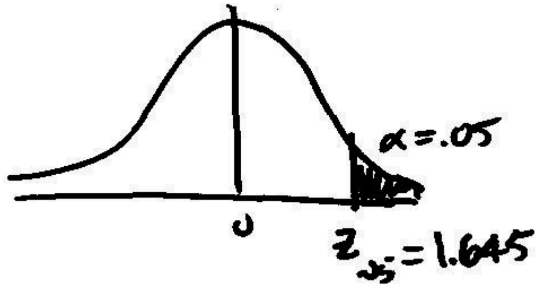
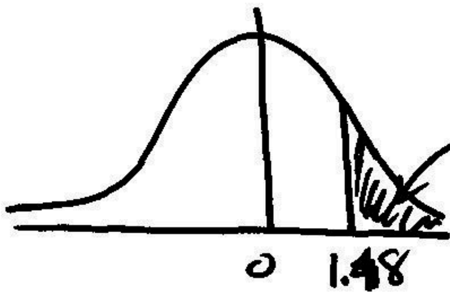


Test stat $= \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{6.7 - 6}{2/\sqrt{18}} = 1.48$

we fail to reject H_0, since T.S. $< z_\alpha$



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Alternative way of expressing the rejection rule for <u>any</u> hypothesis test:

Reject H_0 if p-value $< \alpha$.