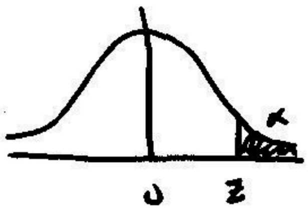
Stat 563
4-16-26

①

Additional comments about the example from last time:

$H_0: \mu \le \mu_0$
$H_1: \mu > \mu_0$



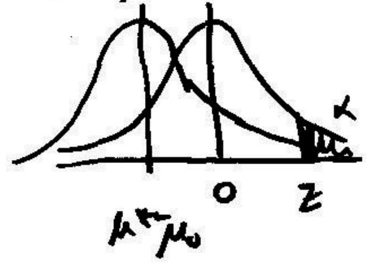
Our solution was to reject H_0 when

$$\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \ge z$$

What if the true mean was actually $\mu^* < \mu_0$?

Then $E[\bar{x}] = \mu^*$

So $E[\bar{x}-\mu_0] = \mu^* - \mu_0 < 0$



②

So the new tail area to the right of z is $< \alpha$.

Now $\sup_{\Omega_0} P(\Lambda \le c) = \alpha$

Defn: The power function of a test is

$$\pi(\theta) = P(\text{Reject } H_0 \mid \theta) \quad \text{for } \theta \in \Omega_0^c$$

Power = 1 − Prob.(Type II error)

Note: this is a function of θ

③

Defn. Let C be a class of hypothesis tests of size α for

$H_0: \theta \in \Omega_0$

$H_1: \theta \notin \Omega_0$

$\alpha = \text{Prob(Type I)}$
= "size"
= "level of significance"

Then a test in C is (UMP) uniformly most powerful if its power function $\pi(\theta)$ is greater than or equal to $\pi'(\theta)$, the power function for any other test in C, $\forall\, \theta \in \Omega_0^c$

④

Neyman-Pearson Theorem

Suppose we are testing $H_0: \theta = \theta_0$
$H_1: \theta = \theta_1$

$$\left(\text{i.e. } \Omega = \{\theta_0, \theta_1\},\ \Omega_0 = \{\theta_0\}\right)$$

And our rejection region is

$$R = \{\vec{x} : \Lambda \le c\}$$

and c is chosen so that $P_{H_0}(\vec{x} \in R) = \alpha$.

Then this test is MP.

⑤

Proof: Let A be the rejection region for any other test of size α

We need to show

$$P(\vec{x} \in R \mid \theta_1) \geq P(\vec{x} \in A \mid \theta_1),$$

i.e. $$\int_R f(\vec{x} \mid \theta_1)\, d\vec{x} \geq \int_A f(\vec{x} \mid \theta_1)\, d\vec{x}$$

$$= \int_R L(\theta_1)\, d\vec{x} \geq \int_A L(\theta_1)\, d\vec{x}$$

⑥

Consider $$\int_R L(\theta_1)\, d\vec{x} - \int_A L(\theta_1)\, d\vec{x} = \star$$

We need to show that $\star \geq 0$

$$\star = \cancel{\int_{R \cap A} L(\theta_1)\, d\vec{x}} + \int_{R \cap A^c} L(\theta_1)\, d\vec{x} - \left[\cancel{\int_{A \cap R} L(\theta_1)\, d\vec{x}} + \int_{A \cap R^c} L(\theta_1)\, d\vec{x} \right]$$

$$= \int_{R \cap A^c} L(\theta_1)\, d\vec{x} - \int_{A \cap R^c} L(\theta_1)\, d\vec{x}$$

(7)

Suppose $\vec{x} \in R$. Then $\Lambda \le c$

$$\frac{L(\theta_0)}{L(\theta_1)} \le c$$

$$L(\theta_1) \ge \frac{1}{c} L(\theta_0)$$

So $$\int_{R \cap A^c} L(\theta_1)\,d\vec{x} \ge \frac{1}{c}\int_{R \cap A^c} L(\theta_0)\,d\vec{x}$$

Suppose $\vec{x} \notin R$. Then $\Lambda > c$

So $L(\theta_1) < \frac{1}{c} L(\theta_0)$

(8)

Now $$\int_{A \cap R^c} L(\theta_1)\,d\vec{x} < \frac{1}{c}\int_{A \cap R^c} L(\theta_0)\,d\vec{x}$$

So $$\star \ge \frac{1}{c}\int_{R \cap A^c} L(\theta_0)\,d\vec{x} - \frac{1}{c}\int_{A \cap R^c} L(\theta_0)\,d\vec{x}$$

$$= \frac{1}{c}\left[\int_{R \cap A^c} L(\theta_0)\,d\vec{x} + \int_{R \cap A} L(\theta_0)\,d\vec{x} - \int_{R \cap A} L(\theta_0)\,d\vec{x} - \int_{A \cap R^c} L(\theta_0)\,d\vec{x}\right]$$

(9)

$$= \frac{1}{c}\left[\int_R L(\theta_0)\,d\vec{x} - \int_A L(\theta_0)\,d\vec{x}\right]$$

$$= \frac{1}{c}\left[\int_R f(\vec{x}\,|\,\theta_0)\,d\vec{x} - \int_A f(\vec{x}\,|\,\theta_0)\,d\vec{x}\right]$$

$$= \frac{1}{c}\left[P(\vec{x}\in R\,|\,\theta_0) - P(\vec{x}\in A\,|\,\theta_0)\right]$$

$$= \frac{1}{c}\left[\alpha - \alpha\right] = 0$$

$\therefore \bigstar \geq 0$

(10)

Recap: For testing $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$, the L.R.T. is most powerful.

Consider the simple null vs. composite alternative hypotheses.

$H_0: \theta = \theta_0$

$H_1: \theta > \theta_0$

Neyman-Pearson Thm says that LRT is most powerful for $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$ for any $\theta_1 > \theta_0$

(11)

If it happens that the structure of the test does not involve the actual value of θ_1, then the LRT would be UMP.

Example: $X_1, \ldots, X_{10} \sim$ iid Poisson(θ)

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} \quad x = 0,1,2,\ldots$$

$H_0: \theta = .1$

$H_1: \theta > .1$

Consider $H_0: \theta = .1$
$H_1: \theta = \theta_1$ where $\theta_1 > .1$

$$L(\theta) = \frac{e^{-10\theta}\,\theta^{\Sigma x_i}}{\Pi(x_i!)}$$

(12)

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\dfrac{e^{-10(.1)}(.1)^{\Sigma x_i}}{\cancel{\Pi(x_i!)}}}{\dfrac{e^{-10(\theta_1)}\,\theta_1^{\Sigma x_i}}{\cancel{\Pi(x_i!)}}}$$

$$\Lambda = \frac{e^{-1}(.1)^{\Sigma x_i}}{e^{-10\theta_1}\,\theta_1^{\Sigma x_i}} \overset{\text{set}}{\leq} c$$

$$\frac{1}{(10\theta_1)^{\Sigma x_i}} \leq c'$$

(13)

$$\sum x_i \ln\left(\frac{1}{10\theta_1}\right) \le c''$$

$$\sum x_i \ge c'''$$

Under H_0, $\sum_{i=1}^{10} x_i \sim \text{Poisson}(1)$

Choose c''' so that $P(\sum x_i \ge c''') = \alpha$

Note that this no longer involves θ_1, so we have a test that is most powerful $\forall\, \theta_1 \notin \Omega_0$, so it is UMP.

(14)

Table of Poisson(1) distribution:

k	$P(X \ge k)$
0	1
1	.632
2	.264
3	.08
4	.019

Question: how do you construct a test with $\alpha = .05$?

Let $p = \dfrac{.05 - .019}{.08 - .019} = \dfrac{31}{61}$

Decision rule: If $\sum x_i \ge 4$, Reject H_0

If $\sum x_i \le 2$, Fail to reject H_0.

If $\sum x_i = 3$, flip a coin whose "heads" prob is p.

Reject H_0 if you get "heads" (15)

$$P_{H_0}(\text{Reject } H_0) = P_{H_0}(\Sigma x_i \geq 4) + p\,P(\Sigma x_i = 3)$$

$$= .019 + \frac{31}{61}(.08 - .019) = .05$$