Stat 563
4-7-26

The Rao-Blackwell Theorem

①

Let $X_1, \ldots, X_n \sim$ iid $f(x|\theta)$

Let T be a sufficient statistic for θ.

Let W be an unbiased estimator of $\tau(\theta)$.

Let $\phi(T) = E[W|T]$

Then $\phi(T)$ is an unbiased estimator of $\tau(\theta)$

and $V[\phi(T)] \leq V[W]$ $\forall$ θ.

②

<u>Proof</u>: $E[\phi(T)] = E[E[W|T]]$

$= E[W] = \tau(\theta)$

$$V[W] = \underbrace{E[V[W|T]]}_{\geq 0} + \underbrace{V[E[W|T]]}_{V[\phi(T)]}$$

$V[W] \geq V[\phi(T)]$

Also, $f(\vec{x}|T)$ is free of θ, so distribution of $W|T$ is free of θ, as is $\phi(T) = E(W|T)$.

③

Example: $X_1, X_2 \sim iid \ \frac{1}{\theta} e^{-x/\theta} \quad x>0$

Let $W = X_1 \qquad E[W] = \theta \qquad V[W] = \theta^2$

Find a sufficient statistic for θ.

$$L(\theta) = \frac{1}{\theta} e^{-\frac{x_1}{\theta}} \frac{1}{\theta} e^{-\frac{x_2}{\theta}}$$

$$= \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1+x_2)} \cdot 1$$

So $T = X_1 + X_2$ is sufficient for θ.

④

Find $E[W|T]$

Start with the joint distribution of W, T.

$W = X_1$	$X_1 = w$	$X_1 > 0$ $X_2 > 0$
$T = X_1 + X_2$	$X_2 = T - w$	$w > 0$ $t - w > 0$ $t > w$

$$|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$g(w,t) = f(x_1, x_2) \cdot 1 = \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1+x_2)}$$

$$= \frac{1}{\theta^2} e^{-t/\theta} \qquad 0 < w < t$$

$$g(w|t) = \frac{g(w,t)}{h(t)} = \frac{\frac{1}{\theta^2} e^{-t/\theta}}{\frac{1}{\theta^2} t e^{-t/\theta}}$$ ⑤

Because $T \sim \text{Gamma}(\alpha = 2, \beta = \theta)$

$$= \frac{1}{t}, \quad 0 < w < t$$

That is, $W|T \sim \text{Unif}(0, T)$

So $E[W|T] = \frac{0+T}{2} = \frac{T}{2} = \frac{X_1 + X_2}{2} = \bar{X}$

Note $E[\bar{X}] = \theta$ and $V[\bar{X}] = \frac{\theta^2}{2}$

⑥

A simpler approach might be to

① Find a sufficient statistic for θ, called T

② Find a function of T that is an unbiased estimator of θ.

Defn: W is a minimum variance unbiased (MVUE) estimator of θ if $E[W] = \theta$ and its variance is less than or equal to the variance of any other unbiased estimator of θ, $\forall \theta$.

Theorem: If W is a MVUE of θ, then W is unique.

Proof: Let W' also be MVUE of θ.

Let $W^* = \frac{1}{2}(W + W')$

$$E[W^*] = \frac{1}{2}[E(W) + E(W')] = \theta$$

$$V[W^*] = \frac{1}{4}[\underline{V[W] + V[W']} + 2\,Cov(W,W')]$$

$$= \frac{1}{4}[2V[W] + 2\,Cov(W,W')]$$

⑧

$$V[W^*] = \frac{1}{2}V[W] + \frac{1}{2}Cov(W,W')$$

Recall $\rho^2 = \dfrac{Cov^2(W,W')}{V[W]V[W']} \leq 1$

So $Cov^2(W,W') \leq (V[W])^2$

$$|Cov(W,W')| \leq V[W] \quad \bigstar$$

Now $V[W^*] \leq \frac{1}{2}V[W] + \frac{1}{2}V[W]$

$$V[W^*] \leq V[W]$$

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If, for any θ, $V[W^*] < V[W]$,
our assumption that W was MVUE
would be violated.

$$\therefore V[W^*] = V[W] \quad \forall \theta$$

Equivalently, $\rho = Cor(W, W') = 1$

Cauchy-Schwarz says $W' = aW + b$

By $\bigstar$, $Cov(W, W') = V[W]$

(10)

$$Cov(W, W') = Cov(W, aW + b)$$
$$= a\, Cov(W, W) + Cov(W, b)$$
$$= a\, V[W] + 0$$
$$\therefore a = 1$$

So $W' = W + b$

$$E[W'] = E[W] + b$$
$$\theta = \theta + b \quad \therefore b = 0$$
$$\therefore W = W'$$

(11)

The Lehmann-Scheffé Theorem

Let T be a complete sufficient statistic for θ. If there exists W, a function of T, that is an unbiased estimator of $\tau(\theta)$, then W is the unique MVUE of $\tau(\theta)$.

Proof: Assume W exists.

Let W' be another unbiased estimator of $\tau(\theta)$.

(12)

By the Rao-Blackwell Theorem,

$E[W' | T]$ will be an unbiased estimator of $\tau(\theta)$ and its variance will be $\leq V[W']$.

Also note that $E[W' | T]$ is a function of T.

Let $\varphi(T)$ and $\psi(T)$ be functions of T that are both unbiased estimators of $\tau(\theta)$.

$$E[\varphi(T) - \psi(T)] = E[\varphi(T)] - E[\psi(T)]$$
$$= \tau(\theta) - \tau(\theta) = 0 \quad \forall \theta$$

By completeness, this implies
$$\varphi(T) - \psi(T) = 0$$

So W and $E[W'|T]$ must be equal.

Then $V[W] = V[E[W'|T]] \leq V[W']$

So W is MVUE and is unique.