

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be unbiased estimators of θ .

Stat 523
4-2-26

①

Defn: The relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$ is

$$\frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$$

Defn: The efficiency of $\hat{\theta}_1$ is $\frac{\text{CRLB}}{V(\hat{\theta}_1)}$

Defn: $\hat{\theta}_1$ is efficient if the efficiency is 1.

Defn: $\hat{\theta}$ is asymptotically unbiased

$$\text{if } \lim_{n \rightarrow \infty} \text{Bias}(\hat{\theta}) = 0$$

Defn: If $\hat{\theta}_1$ and $\hat{\theta}_2$ are asymptotically unbiased estimators of θ , then the asymptotic relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$

$$\text{is } \lim_{n \rightarrow \infty} \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}.$$

②

Defn: If $\hat{\theta}_n$ is asymptotically unbiased,

(3)

then its asymptotic efficiency is

$$\lim_{n \rightarrow \infty} \frac{\text{CRLB}}{V(\hat{\theta}_n)}$$

Example: $X_1, \dots, X_n \sim \text{iid Exp}(\lambda)$

Compare $\hat{\lambda}_{\text{MLE}}$ to an unbiased version of it.

$$L(\lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum x_i}$$

(4)

$$\ell(\lambda) = \ln L(\lambda) = n \ln \lambda - \lambda \sum x_i$$

$$\ell'(\lambda) = \frac{n}{\lambda} - \sum x_i \stackrel{\text{set}}{=} 0$$

$$\hat{\lambda}_{\text{MLE}} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

$$E[\hat{\lambda}_{\text{MLE}}] = E\left[\frac{n}{\sum x_i}\right] = E\left[\frac{n}{Y}\right] = n E[Y^{-1}]$$

$$Y = \sum x_i \sim \text{Gamma}(\alpha = n, \beta = \frac{1}{\lambda})$$

$$E[Y^{-1}] = \beta^{-1} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} = \lambda \frac{(n-2)!}{(n-1)!} = \frac{\lambda}{n-1} \quad (5)$$

$$E[\hat{\lambda}_{MLE}] = n E[Y^{-1}] = \frac{n}{n-1} \lambda$$

$$\text{Bias}(\hat{\lambda}_{MLE}) = \frac{n}{n-1} \lambda - \lambda = \frac{\lambda}{n-1}$$

$\therefore \hat{\lambda}_{MLE}$ is biased, but asymptotically unbiased.

$$\text{Define } \hat{\lambda}_u = \frac{n-1}{n} \hat{\lambda}_{MLE} = \frac{n-1}{n} \frac{1}{\sum x_i} = \frac{n-1}{\sum x_i}$$

$$\begin{aligned} V(\hat{\lambda}_{MLE}) &= V\left(\frac{1}{\sum x_i}\right) = V\left(\frac{n}{Y}\right) \quad (6) \\ &= n^2 V[Y^{-1}] \\ &= n^2 \left[E[Y^{-2}] - \underbrace{(E[Y^{-1}])^2}_{\frac{\lambda^2}{(n-1)^2}} \right] \\ &\quad \downarrow \\ &= n^2 \frac{\beta^{-2} \Gamma(\alpha-2)}{\Gamma(\alpha)} \\ &= n^2 \frac{\lambda^2 (n-3)!}{(n-1)!} \\ &= \frac{\lambda^2}{(n-1)(n-2)} \end{aligned}$$

(7)

$$V(\hat{\lambda}_{MLE}) = n^2 \left[\frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2} \right]$$

$$= \frac{n^2 \lambda^2}{(n-1)^2 (n-2)}$$

$$V(\hat{\lambda}_U) = V\left(\frac{n-1}{n} \hat{\lambda}_{MLE}\right) = \left(\frac{n-1}{n}\right)^2 \frac{n^2 \lambda^2}{(n-1)^2 (n-2)}$$

$$= \frac{\lambda^2}{n-2}$$

The asymptotic relative efficiency of $\hat{\lambda}_U$ to $\hat{\lambda}_{MLE}$

$$(5) \quad \lim_{n \rightarrow \infty} \frac{\frac{n^2 \lambda^2}{(n-1)^2 (n-2)}}{\frac{\lambda^2}{n-2}} = \lim_{n \rightarrow \infty} \frac{n^2}{(n-1)^2} = 1 \quad (8)$$

Find the Cramér-Rao Lower Bound (CRLB):

$$f(x) = \lambda e^{-\lambda x}$$

$$\ln f(x) = \ln \lambda - \lambda x$$

$$\frac{\partial \ln f}{\partial \lambda} = \frac{1}{\lambda} - x$$

$$\frac{\partial^2 \ln f}{\partial \lambda^2} = -\frac{1}{\lambda^2}$$

(11)

$$\text{MSE}(\hat{\theta}) = \text{B}^2(\hat{\theta}) + \text{V}(\hat{\theta})$$

$$\text{MSE}(\hat{\lambda}_u) = \text{V}(\hat{\lambda}_u) = \frac{\lambda^2}{n-2}$$

$$\text{MSE}(\hat{\lambda}_{MLE}) = \left(\frac{\lambda}{n-1}\right)^2 + \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$= \lambda^2 \frac{(n-2) + n^2}{(n-1)^2(n-2)}$$

$$= \lambda^2 \frac{n^2+n-2}{(n-1)^2(n-2)} = \lambda^2 \frac{(n+2)}{(n-1)(n-2)}$$

$$\therefore \text{MSE}(\hat{\lambda}_{MLE}) > \text{MSE}(\hat{\lambda}_u)$$

7.12 Let X_1, \dots, X_n be a random sample from a population with pmf

$$P_\theta(X = x) = \theta^x(1 - \theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \leq \theta \leq \frac{1}{2}.$$

- (a) Find the method of moments estimator and MLE of θ .
- (b) Find the mean squared errors of each of the estimators.
- (c) Which estimator is preferred? Justify your choice.

7.48 Suppose that $X_i, i = 1, \dots, n$, are iid Bernoulli(p).

- (a) Show that the variance of the MLE of p attains the Cramér–Rao Lower Bound.
- (b) For $n \geq 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4 , and use this fact to find the best unbiased estimator of p^4 .

7.59 Let X_1, \dots, X_n be iid $n(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p , where p is a known positive constant, not necessarily an integer.