

Stat 5263
3-31-26
Defn: Let X have pdf or pmf $f(x|\theta)$.
The score function for X is $\frac{\partial}{\partial \theta} \ln f(x|\theta)$ ①

Consider $\frac{\partial}{\partial \theta} \ln f(X|\theta)$ as a random variable
that is a function of X .

Find its expectation and variance.

$$E\left[\frac{\partial}{\partial \theta} \ln f(X|\theta)\right] = \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \ln f(x|\theta) f(x|\theta) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{f(x|\theta)} \frac{\partial}{\partial \theta} f(x|\theta) f(x|\theta) dx \quad \text{②}$$

$$= \frac{\partial}{\partial \theta} \underbrace{\int_{-\infty}^{\infty} f(x|\theta) dx}_1$$

* subject to some regularity conditions

$$= 0$$

$$V\left[\frac{\partial}{\partial \theta} \ln f(X|\theta)\right] = E\left[\left[\frac{\partial}{\partial \theta} \ln f(X|\theta)\right]^2\right]$$

Defn: This is the Fisher Information $I(\theta)$

There is an alternate form that is more common. (3)

From the previous derivation,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \ln f(x|\theta) f(x|\theta) dx = 0$$

Take $\frac{\partial}{\partial \theta}$ of both sides

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial \theta} \ln f(x|\theta) f(x|\theta) \right] dx = 0$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \theta} \ln f(x|\theta) \frac{\partial}{\partial \theta} f(x|\theta) + f(x|\theta) \frac{\partial^2}{\partial \theta^2} \ln f(x|\theta) \right] dx \quad (4)$$

$$= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \theta} \ln f(x|\theta) \right) \left(\frac{\frac{\partial}{\partial \theta} f(x|\theta)}{f(x|\theta)} \right) f(x|\theta) dx$$

$$+ \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \theta^2} \ln f(x|\theta) f(x|\theta) dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \theta} \ln f(x|\theta) \right)^2 f(x|\theta) dx + E \left[\frac{\partial^2}{\partial \theta^2} \ln f(x|\theta) \right]$$

$$\text{So } 0 = I(\theta) + E\left[\frac{\partial^2}{\partial \theta^2} \ln f(X|\theta)\right] \quad (5)$$

$$\therefore I(\theta) = - E\left[\frac{\partial^2}{\partial \theta^2} \ln f(X|\theta)\right]$$

Let $X_1, \dots, X_n \sim \text{iid } f(x|\theta)$

Let W be a function of X_1, \dots, X_n (free of θ)

Suppose $E[W] = k(\theta)$

$$k(\theta) = E[W] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} w \underbrace{\prod_{i=1}^n f(x_i|\theta)}_{L(\theta)} dx_1 \dots dx_n \quad (6)$$

$$k'(\theta) = \frac{\partial}{\partial \theta} E[W] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} w L'(\theta) dx_1 \dots dx_n$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} w \underbrace{\left(\frac{L'(\theta)}{L(\theta)}\right)}_Z L(\theta) dx_1 \dots dx_n$$

(7)

$$= E[WZ]$$

So far: $K(\theta) = E[W]$

$$K'(\theta) = E[WZ]$$

$$\begin{aligned} Z &= \frac{L'(\theta)}{L(\theta)} = \frac{\partial}{\partial \theta} \ln L(\theta) \\ &= \frac{\partial}{\partial \theta} \ln \prod_{i=1}^n f(x_i | \theta) \end{aligned}$$

$$= \frac{\partial}{\partial \theta} \sum_{i=1}^n \ln f(x_i | \theta)$$

$$= \sum_{i=1}^n \frac{\partial}{\partial \theta} \ln f(x_i | \theta)$$

$$E[Z] = \sum_{i=1}^n E\left[\frac{\partial}{\partial \theta} \ln f(x_i | \theta)\right]$$

$$= 0 \quad (\text{Expected value of score function was 0})$$

(8)

This means that

(9)

$$\begin{aligned}\text{Cov}(W, Z) &= E[WZ] - E[W]E[Z] \\ &= E[WZ]\end{aligned}$$

$$\begin{aligned}\text{Also, } V[Z] &= V\left[\sum_{i=1}^n \frac{\partial}{\partial \theta} \ln f(X_i|\theta)\right] \\ &= \sum_{i=1}^n V\left[\frac{\partial}{\partial \theta} \ln f(X_i|\theta)\right] \text{ by independence} \\ &= n I(\theta)\end{aligned}$$

$$\text{Corr}(W, Z) = \frac{\text{Cov}(W, Z)}{\sqrt{V(W)V(Z)}}$$

(10)

$$\frac{\text{Cov}^2(W, Z)}{V(W)V(Z)} \leq 1 \quad \text{Since } -1 \leq \rho \leq 1$$

$$\frac{[k'(\theta)]^2}{V(W) n I(\theta)} \leq 1$$

$$V[W] \geq \frac{[k'(\theta)]^2}{n I(\theta)} \quad (k(\theta) = E[W]) \quad (1)$$

This is called the Cramer-Rao Inequality

Suppose that W is an unbiased estimator of θ .

$$\text{Then } k(\theta) = E[W] = \theta \quad \text{and } k'(\theta) = 1$$

$$\text{So } V[W] \geq \frac{1}{n I(\theta)} \quad \left. \begin{array}{l} \text{C.R.L.B.} \\ \text{This is the lower bound} \\ \text{on the variance of an} \\ \text{unbiased estimator.} \end{array} \right\}$$

Example: $X_1, \dots, X_n \sim \text{iid}$ $f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}$ (2)
 $x > 0$

Know that $\hat{\theta}_{\text{MSE}} = \bar{x}$

$$E[\hat{\theta}] = E[\bar{x}] = \mu = \theta \quad \text{So } \hat{\theta} \text{ is unbiased}$$

$$V[\hat{\theta}] = V[\bar{x}] = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$$

Find the C.R.L.B.

$$\ln f(x|\theta) = -\ln \theta - \frac{x}{\theta}$$

(13)

$$\frac{\partial}{\partial \theta} \ln f(x|\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}$$

$$\frac{\partial^2}{\partial \theta^2} \ln f(x|\theta) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$I(\theta) = -E\left[\frac{1}{\theta^2} - \frac{2x}{\theta^3}\right] = -\left[\frac{1}{\theta^2} - \frac{2\theta}{\theta^3}\right]$$

$$C.R.L.B. = \frac{1}{nI(\theta)} = \frac{1}{n \cdot \frac{1}{\theta}} = \frac{\theta}{n}$$

(14)

Defn. The efficiency of an estimator is $\frac{CRLB}{V(\hat{\theta})}$

If the efficiency = 1, then the estimator is efficient