Stat 561
12-2-25

Continuous Case: Assume every component of the random vector is continuous

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Defn. $f(x_1, \ldots, x_n)$ is a **joint probability density function** if, $\forall A \subset \mathbb{R}^n$

$$P[\vec{X} \in A] = \int\int_A \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

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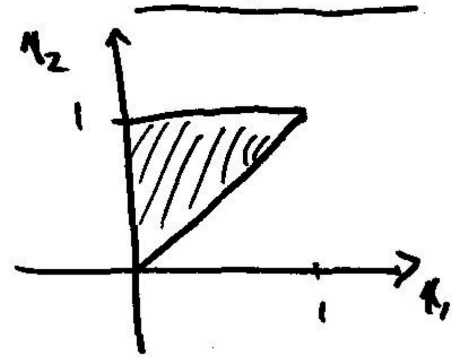
Example: $f(x_1, x_2) = 8x_1x_2 \quad \underline{0 < x_1 < x_2 < 1}$



Check to see if $f(x_1, x_2)$ integrates to 1.

$$\underbrace{\int_0^1\int_0^{x_2} 8x_1x_2\, dx_1\, dx_2}_{} \quad \text{OR} \quad \int_0^1\int_{x_1}^1 8x_1x_2\, dx_2\, dx_1$$

$$8\int_0^1 x_2 \frac{x_1^2}{2}\Big|_{x_1=0}^{x_2} dx_2 = 8\int_0^1 \frac{x_2^3}{2}\, dx_2 = x_2^4\Big|_0^1 = 1 \checkmark$$

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The marginal prob. density fn for X_1 is

$$f_1(x_1) = \int_{x_1}^{1} f(x_1, x_2)\, dx_2$$

$$= \int_{x_1}^{1} 8 x_1 x_2\, dx_2 = 8 x_1 \left. \frac{x_2^2}{2} \right|_{x_2 = x_1}^{1}$$

$$= 4x_1 - 4x_1^3$$

$$f_1(x_1) = 4x_1 - 4x_1^3, \quad 0 < x_1 < 1$$

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The marginal prob. density fn. for X_2 is

$$f_2(x_2) = \int_{0}^{x_2} 8 x_1 x_2\, dx_1$$

$$= 8 x_2 \left. \frac{x_1^2}{2} \right|_{x_1 = 0}^{x_2} = 4x_2^3$$

$$f_2(x_2) = 4x_2^3 \qquad 0 < x_2 < 1$$

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Let $g(X_1, X_2, \ldots, X_n)$ be a new random variable.

<u>Defn.</u> $$E[g(\vec{X})] = \begin{cases} \sum \cdots \sum_{\text{all } \vec{x}} g(\vec{x})p(\vec{x}) \\ \int \cdots \int_{\mathbb{R}^n} g(\vec{x})f(\vec{x})\,d\vec{x} \end{cases}$$

In our example, find $E[X_1 + X_2]$

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Method 1: $$E[X_1+X_2] = \int_0^1 \int_0^{x_2} (x_1+x_2)\, 8x_1 x_2 \, dx_1\, dx_2$$

$$= \int_0^1 \int_0^{x_2} (8x_1^2 x_2 + 8x_1 x_2^2)\, dx_1\, dx_2$$

$$= \int_0^1 \left(\frac{8x_1^3 x_2}{3} + \frac{8x_1^2 x_2^2}{2} \right) \Bigg|_{x_1=0}^{x_2} dx_2$$

$$= \int_0^1 \frac{8x_2^4}{3} + \frac{8x_2^4}{2} \, dx_2 = \frac{20}{3} \int_0^1 x_2^4 \, dx_2$$

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$$= \frac{20}{3} \frac{x_2^5}{5} \Big|_0^1 = \frac{4}{3}.$$

Method 2: $E[X_1 + X_2] = E[X_1] + E[X_2]$

$$E[X_1] = \int_0^1 x_1 (4x_1 - 4x_1^3)\, dx_1 = 8/15$$

$$E[X_2] = \int_0^1 x_2\, 4x_2^3\, dx_2 = \frac{4}{5}$$

So $E[X_1 + X_2] = 8/15 + 4/5 = \frac{4}{3}.$

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Conditional distributions

Defn: $\begin{cases} f(x_2 | x_1) = \dfrac{f(x_1, x_2)}{f_1(x_1)} \\ \\ p(x_2 | x_1) = \dfrac{p(x_1, x_2)}{p_1(x_1)} \end{cases}$

Note: $\displaystyle\int_{\text{all } x_2} f(x_2 | x_1)\, dx_2 = \int_{\text{all } x_2} \frac{f(x_1, x_2)}{f_1(x_1)}\, dx_2$

$$= \frac{1}{f_1(x_1)} \underbrace{\int_{\text{all } x_2} f(x_1, x_2)\, dx_2}_{f_1(x_1)} = 1 \quad \checkmark$$

Let $g(X_2)$ be a function of X_2.

Defn: $$E[g(X_2) \mid x_1] = \begin{cases} \displaystyle\sum_{\text{all } x_2} g(x_2)\, p(x_2 \mid x_1) \\ \displaystyle\int_{\text{all } x_2} g(x_2)\, f(x_2 \mid x_1)\, dx_2 \end{cases}$$

Note: $E[g(X_2)]$ = number

$E[g(X_2) \mid x_1]$ = function of x_1

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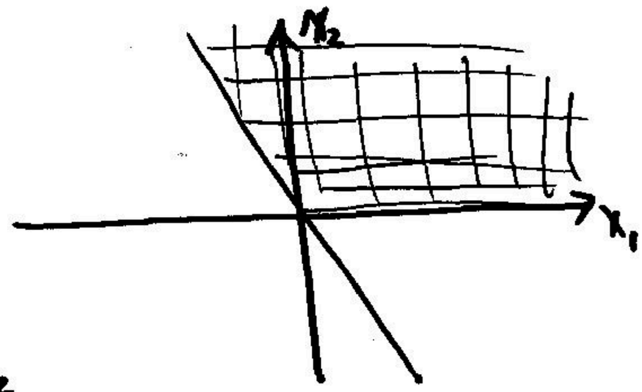
Example: $f(x_1, x_2) = \frac{1}{2} e^{-(x_1 + x_2)}$, $\quad x_2 > 0$, $\quad x_2 > -2x_1$

Find $f(x_2 \mid x_1)$



$$f_1(x_1) = \int_{\text{all } x_2} f(x_1, x_2)\, dx_2$$

$$= \begin{cases} \displaystyle\int_0^\infty \frac{1}{2} e^{-(x_1 + x_2)}\, dx_2 & x_1 \geq 0 \\ \displaystyle\int_{-2x_1}^\infty \frac{1}{2} e^{-(x_1 + x_2)}\, dx_2 & x_1 < 0 \end{cases}$$

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$$= \begin{cases} \frac{1}{2}e^{-x_1}\left[-e^{-x_2}\right]_{x_2=0}^{\infty} & x_1 \geq 0 \\ = 0 - \left(-\frac{1}{2}e^{-x_1}\right) = \frac{1}{2}e^{-x_1} & \\ \hline \frac{1}{2}e^{-x_1}\left[-e^{-x_2}\right]_{x_2=-2x_1}^{\infty} & x_1 < 0 \\ = 0 + \frac{1}{2}e^{-x_1}e^{2x_1} = \frac{1}{2}e^{x_1} & \end{cases}$$

$$= \frac{1}{2}e^{-|x_1|} \qquad -\infty < x_1 < \infty$$

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$$f(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

$$= \frac{\frac{1}{2}e^{-(x_1+x_2)}}{\frac{1}{2}e^{-|x_1|}}$$

$$= e^{|x_1| - x_1 - x_2} \qquad \begin{matrix} x_2 > 0 \\ x_2 > -2x_1 \end{matrix}$$

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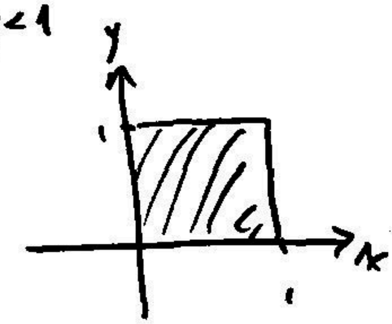
New example:

$$f(x,y) = 10xy^4 \quad 0<x<1,\ 0<y<1$$



$$\int_0^1\int_0^1 10xy^4\,dx\,dy$$

$$= \int_0^1 10y^4 \left.\frac{x^2}{2}\right|_{x=0}^1 dy$$

$$= \int_0^1 10y^4 \cdot \tfrac{1}{2}\,dy = 5\left.\frac{y^5}{5}\right|_0^1 = 1 \ \checkmark$$

Find $f(y|x)$

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$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 10xy^4\,dy$$

$$= 10x\left.\frac{y^5}{5}\right|_0^1 = 2x$$

$$f_X(x) = 2x,\quad 0<x<1$$

Now, $f(y|x) = \dfrac{10xy^4}{2x} = 5y^4,\quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$

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<u>Defn.</u>: The random variables X and Y are <u>independent</u> if $f(x,y) = f_X(x) f_Y(y)$ $\forall (x,y)$

<u>Factorization Theorem</u>: The random variables X & Y are independent <u>if and only if</u> there exist functions $g(x)$ and $h(y)$ such that
$$f(x,y) = g(x)h(y) \quad \forall (x,y).$$

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<u>Proof</u>: $\Rightarrow$: trivial, based on definition

$\Leftarrow$: Suppose $\exists$ $g(x)$ & $h(y)$ such that $f(x,y) = g(x)h(y)$

Then
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_{-\infty}^{\infty} g(x)h(y)\,dy = g(x)\underbrace{\int_{-\infty}^{\infty} h(y)\,dy}_{k}$$

Now $f_X(x) = k\,g(x)$

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Also, $1 = \int_{-\infty}^{\infty} f_X(x)\,dx$

$$= \int_{-\infty}^{\infty} k\,g(x)\,dx = k \int_{-\infty}^{\infty} g(x)\,dx$$

So $\int_{-\infty}^{\infty} g(x)\,dx = \frac{1}{k}$

And $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} g(x)h(y)\,dx$

$$= h(y) \int_{-\infty}^{\infty} g(x)\,dx = \frac{1}{k} h(y)$$

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$$\therefore\ f(x,y) = g(x)h(y)$$

$$= \underline{k\,g(x)}\ \underline{\tfrac{1}{k} h(y)}$$

$$= f_X(x)\,f_Y(y),$$ so X & Y are independent

Examples:

$f(x,y) = c(x+y) \quad 0<x<1 \quad 0<y<1$

Not independent

$$f(x,y) = ce^{-(x+y)} \quad x>0,\ y>0$$

Independent

$$f(x,y) = cxy \quad 0<x<y<1$$

$$= cxy\, I_{\{0<x<y<1\}}(x,y)$$

Not independent

Exam 4 topics

Continuous distributions:

Normal
Lognormal
Chi-squared (Gamma)
Beta
Cauchy
Uniform
Laplace

Exponential families

Location and scale parameters