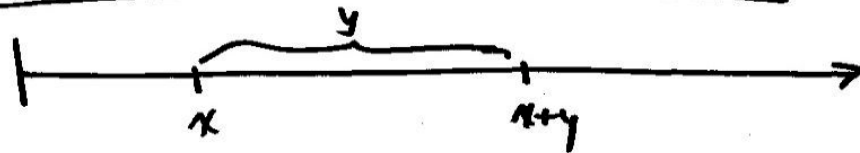


A property of the exponential distribution

Stat 501  
11-20-25



(1)

$$P(X > x+y | X > x) = \frac{P(X > x+y \cap X > x)}{P(X > x)}$$

↑  
waiting time until 1<sup>st</sup> occurrence

$$= \frac{P(X > x+y)}{P(X > x)}$$

$$= \frac{1 - F_X(x+y)}{1 - F_X(x)}$$

$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}}$$

$$= e^{-\lambda y} = 1 - F_X(y) = P(X > y)$$

$$\left. \begin{aligned} f_X(x) &= \lambda e^{-\lambda x} \\ F_X(x) &= 1 - e^{-\lambda x} \end{aligned} \right\}$$

"memoryless"  
or  
"forgetfulness"

(2)

The Beta function is

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

The Beta density function is

$$f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad 0 < x < 1$$

③

How is this related to the gamma function?

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \left( \int_0^\infty s^{\alpha-1} e^{-s} ds \right) \left( \int_0^\infty t^{\beta-1} e^{-t} dt \right) \\ &= \int_0^\infty \int_0^\infty s^{\alpha-1} t^{\beta-1} e^{-(s+t)} ds dt\end{aligned}$$

$$\text{let } s = uv, \quad t = (1-u)v$$

$$u = \frac{s}{v} \quad t = \left(1 - \frac{s}{v}\right)v = v - s$$

$$\text{so } v = s+t \\ u = \frac{s}{s+t}$$

④

$$J = \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{(s+t)-s}{(s+t)^2} & s(-1)(s+t)^{-2} \\ 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{t}{(s+t)^2} & \frac{-s}{(s+t)^2} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{t}{(s+t)^2} - \frac{-s}{(s+t)^2} = \frac{s+t}{(s+t)^2} = \frac{1}{s+t} = \frac{1}{v}$$

(5)

$$\begin{aligned}
 \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^1 (uv)^{\alpha-1} (1-uv)^{\beta-1} e^{-v} v \, du \, dv \\
 &= \int_0^\infty \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-v} \, du \, dv \\
 &= \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} \, du \int_0^\infty v^{\alpha+\beta-1} e^{-v} \, dv \\
 &= B(\alpha, \beta) \Gamma(\alpha+\beta) \\
 \therefore B(\alpha, \beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}
 \end{aligned}$$

$$E[X^n] = \int_0^1 \frac{x^n}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \, dx$$

(6)

$$= \int_0^1 \frac{x^{n+\alpha-1}}{B(\alpha, \beta)} (1-x)^{\beta-1} \, dx$$

$$= \frac{B(n+\alpha, \beta)}{B(\alpha, \beta)} \underbrace{\int_0^1 \frac{1}{B(n+\alpha, \beta)} x^{n+\alpha-1} (1-x)^{\beta-1} \, dx}_1$$

$$E[X^n] = \frac{B(n+\alpha, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(n+\alpha)\Gamma(\beta)}{\Gamma(n+\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \quad (7)$$

$$\begin{aligned} \text{So } E[X] &= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)} \\ &= \frac{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\alpha+\beta)\Gamma(\alpha+\beta)} \\ &= \frac{\alpha}{\alpha+\beta} \end{aligned}$$

$\Gamma(x) = (x-1)\Gamma(x-1)$

$$\begin{aligned} E[X^2] &= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha)} \\ &= \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} \\ &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \quad (8) \end{aligned}$$

$$\sigma^2 = E[X^2] - \mu^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

The Cauchy distribution

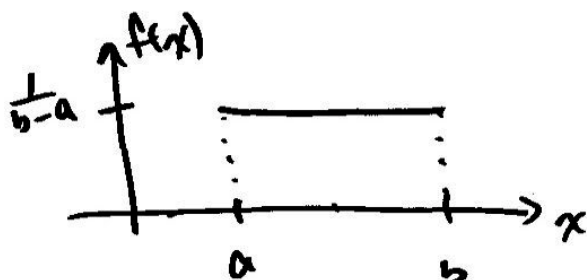
$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2} \quad -\infty < x < \infty$$

Recall:  $\mu$  and  $\sigma^2$  do not exist

$\theta$  is the median

The uniform distribution

$$f_X(x) = \frac{1}{b-a} \quad a < x < b$$



$$\begin{aligned} E[X] &= \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b \\ &= \frac{1}{b-a} \cdot \frac{1}{2} (b^2 - a^2) = \frac{b+a}{2} \end{aligned}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \left. \frac{x^3}{3} \right|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{1}{3} (b^3 - a^3)$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{1}{3} (\cancel{b-a})(b^2 + ab + a^2)$$

$$\sigma^2 = \frac{b^2 + ab + a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2 - 3b^2 - 6ab - 3a^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

(9)

(10)

(11)

$$M_X(t) = \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left. \frac{e^{tx}}{t} \right|_a^b$$

$$= \frac{1}{b-a} \left[ \frac{e^{tb}}{t} - \frac{e^{ta}}{t} \right]$$

$$= \frac{1}{b-a} \frac{e^{tb} - e^{ta}}{t}$$

Note:  $M_X(t)$  is undefined at  $t=0$ , but the limit as  $t \rightarrow 0$  is 1

(12)

Let  $Y$  have a normal distribution

$$N(\mu, \sigma^2)$$

Let  $X = e^Y$

$$y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\begin{aligned} f_X(x) &= g(y) \left| \frac{dy}{dx} \right| \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \cdot \frac{1}{x} \end{aligned}$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} \quad x > 0$$

(13)

This is the lognormal density

$$\begin{aligned} \text{Find } E[X] &= E[e^Y] = E[e^{tY}] \text{ at } t=1 \\ &= M_Y(t) \text{ at } t=1 \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \text{ at } t=1 \\ &= e^{\mu + \frac{1}{2}\sigma^2} \end{aligned}$$

$$E[X^2] = E[(e^Y)^2] = E[e^{2Y}]$$

(14)

$$\begin{aligned} &= E[e^{tY}] \text{ at } t=2 \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \text{ at } t=2 \\ &= e^{2\mu + 2\sigma^2} \end{aligned}$$

$$\begin{aligned} V[X] &= E[X^2] - (E[X])^2 \\ &= e^{2\mu + 2\sigma^2} - (e^{\mu + \frac{1}{2}\sigma^2})^2 \end{aligned}$$

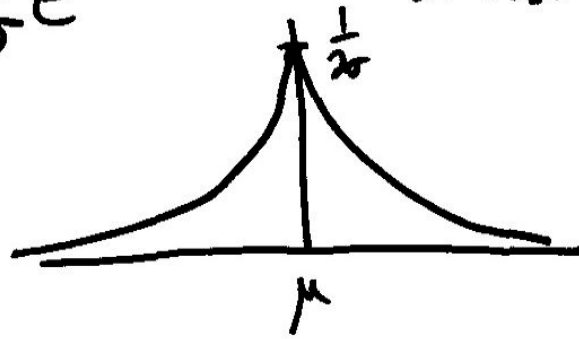
$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$= e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

(15)

The double exponential (Laplace) distribution

$$f_X(x) = \frac{1}{2\sigma} e^{-\left|\frac{x-\mu}{\sigma}\right|} \quad -\infty < x < \infty$$



(16)

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\sigma} e^{-\left|\frac{x-\mu}{\sigma}\right|} dx$$

$$= \int_{-\infty}^{\infty} e^{t(\mu+\sigma y)} \frac{1}{2} e^{-|y|} dy$$

$$\begin{aligned} \text{Let } y &= \frac{x-\mu}{\sigma} \\ dy &= \frac{1}{\sigma} dx \end{aligned}$$

$$x = \mu + \sigma y$$

$$= \int_{-\infty}^0 e^{t(\mu+\sigma y)} \frac{1}{2} e^y dy$$

$$+ \int_0^{\infty} e^{t(\mu+\sigma y)} \frac{1}{2} e^{-y} dy$$

(17)

$$= \frac{1}{2} e^{\mu t} \left[ \int_{-\infty}^0 e^{(\sigma t + 1)y} dy + \int_0^{\infty} e^{(\sigma t - 1)y} dy \right]$$

$$= \frac{1}{2} e^{\mu t} \left[ \frac{e^{(\sigma t + 1)y}}{\sigma t + 1} \Big|_{y=-\infty}^0 + \frac{e^{(\sigma t - 1)y}}{\sigma t - 1} \Big|_{y=0}^{\infty} \right]$$

$$= \frac{1}{2} e^{\mu t} \left[ \frac{1}{\sigma t + 1} - 0 + 0 - \frac{1}{\sigma t - 1} \right]$$

$\uparrow$  true if  $\sigma t + 1 > 0$   $\uparrow$  true if  $\sigma t - 1 < 0$   
 $t > -\frac{1}{\sigma}$   $t < \frac{1}{\sigma}$

(18)

$$= \frac{1}{2} e^{\mu t} \left[ \frac{\sigma t - 1 - (\sigma t + 1)}{(\sigma t + 1)(\sigma t - 1)} \right]$$

$$= \frac{1}{2} e^{\mu t} \frac{-2}{\sigma^2 t^2 - 1}$$

$$M_X(t) = \frac{e^{\mu t}}{1 - \sigma^2 t^2}$$

$$M_X(0) = 1 \quad \checkmark$$

$$M_X'(t) = \frac{(1 - \sigma^2 t^2) \mu e^{\mu t} - e^{\mu t} (-2\sigma^2 t)}{(1 - \sigma^2 t^2)^2}$$

(19)

$$M_X'(t) = \frac{e^{\mu t} (\mu - \mu \sigma^2 t^2 + 2\sigma^2 t)}{(1 - \sigma^2 t^2)^2}$$

$$M_X'(0) = \mu = E[X]$$

You can find  $M_X''(t)$  + get  $E[X^2]$   
+ show  $V[X] = 2\sigma^2$

**3.28** Show that each of the following families is an exponential family.

- (a) normal family with either parameter  $\mu$  or  $\sigma$  known
- (b) gamma family with either parameter  $\alpha$  or  $\beta$  known or both unknown
- (c) beta family with either parameter  $\alpha$  or  $\beta$  known or both unknown
- (d) Poisson family
- (e) negative binomial family with  $r$  known,  $0 < p < 1$

**3.33** For each of the following families:

- (i) Verify that it is an exponential family.
- (ii) Describe the curve on which the  $\theta$  parameter vector lies.
- (iii) Sketch a graph of the curved parameter space.

- (a)  $n(\theta, \theta)$
- (b)  $n(\theta, a\theta^2)$ ,  $a$  known
- (c)  $\text{gamma}(\alpha, 1/\alpha)$
- (d)  $f(x|\theta) = C \exp(-(x - \theta)^4)$ ,  $C$  a normalizing constant

**3.38** Let  $Z$  be a random variable with pdf  $f(z)$ . Define  $z_\alpha$  to be a number that satisfies this relationship:

$$\alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} f(z) dz.$$

Show that if  $X$  is a random variable with pdf  $(1/\sigma)f((x - \mu)/\sigma)$  and  $x_\alpha = \sigma z_\alpha + \mu$ , then  $P(X > x_\alpha) = \alpha$ . (Thus if a table of  $z_\alpha$  values were available, then values of  $x_\alpha$  could be easily computed for any member of the location-scale family.)