Stat 561
11-13-25

①

$$f_T(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1} e^{-t/\beta},$$

$t > 0 \quad (\beta > 0)$

pdf for the Gamma distribution

$$\mu = E[X] = M_X'(0) = \alpha\beta$$

$$E[X^2] = M_X''(0) = \alpha(\alpha+1)\beta^2$$

$$\sigma^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

②

Recall from #3.17 homework:

$$E[X^{\nu}] = \frac{\beta^{\nu}\Gamma(\nu+\alpha)}{\Gamma(\alpha)}$$

Find $\Gamma(.5) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\, dt$

$$= \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\, dt$$

$$= \int_0^\infty e^{-u^2}\, 2\, du$$

Let $u = \sqrt{t}$
$du = \frac{1}{2} t^{-1/2} dt$

③

$$\Gamma(.5) = 2\underbrace{\int_0^\infty e^{-u^2}\,du}_{A} \qquad \Gamma(.5) = 2A$$

$$A^2 = \left(\int_0^\infty e^{-u^2}\,du\right)\left(\int_0^\infty e^{-v^2}\,dv\right)$$

$$= \int_0^\infty\int_0^\infty e^{-u^2}e^{-v^2}\,du\,dv$$

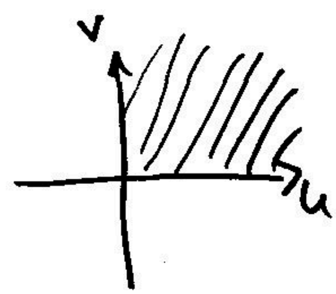
$$= \int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv$$

Let $u = r\cos\theta$
$v = r\sin\theta$

④

$$A^2 = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

Jacobian



Let $t = r^2$
$dt = 2r\,dr$

$$A^2 = \int_0^{\pi/2}\int_0^\infty e^{-t}\,\frac{1}{2}\,dt\;d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\left(-e^{-t}\right)\Big|_0^\infty\right)d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{1}{2}\theta \Big|_0^{\pi/2} = \frac{\pi}{4}$$

$$A^2 = \frac{\pi}{4} \qquad A = \frac{\sqrt{\pi}}{2}$$

$$\underline{\Gamma(.5)} = 2A = \underline{\sqrt{\pi}}$$

Another special case of the gamma distribution:

Let $\alpha = \frac{k}{2}$, where k is a positive integer

and $\beta = 2$

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$$\boxed{f_X(x) = \frac{1}{\Gamma(\frac{k}{2})\, 2^{k/2}}\, x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, \quad x > 0}$$

This is the chi-squared distribution

(χ^2) with k degrees of freedom

$$M_X(t) = \frac{1}{(1-2t)^{k/2}}$$

$$\mu = \alpha\beta = \frac{k}{2} \cdot 2 = k$$

$$\sigma^2 = \alpha\beta^2 = \frac{k}{2} \cdot 4 = 2k$$

⑦

We had $\Gamma(.5) = 2A = 2\int_0^\infty e^{-u^2}\,du$

$$\Gamma(.5) = \sqrt{\pi}$$

So $\displaystyle\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{\sqrt{2}\,\sigma}\,dx$$

Let $u = \frac{1}{\sqrt{2}}\left(\frac{x-\mu}{\sigma}\right)$

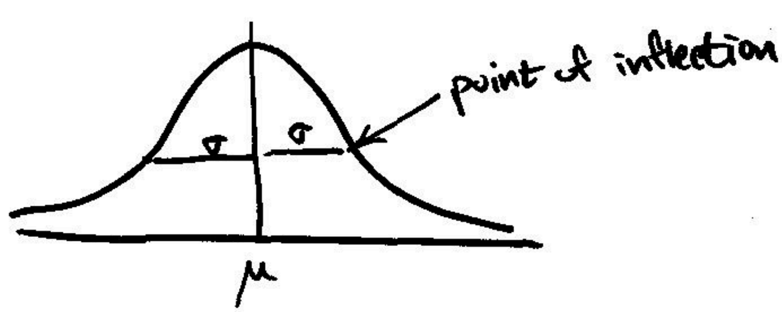
$du = \frac{1}{\sqrt{2}}\frac{1}{\sigma}\,dx$

⑧

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

Define $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty$

to be the normal density function



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$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x\right)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2(\mu + \sigma^2 t)x + \mu^2\right)} dx$$

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$$= e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{(\mu+\sigma^2 t)^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2(\mu+\sigma^2 t)x + (\mu+\sigma^2 t)^2\right)} dx$$

$$= e^{\frac{1}{2\sigma^2}\left(\cancel{-\mu^2} + \cancel{\mu^2} + 2\mu\sigma^2 t + \sigma^4 t^2\right)} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x - (\mu+\sigma^2 t)\right)^2} dx}_{1}$$

(11)

$$M_X(t) = e^{\frac{1}{2}(2\mu t + \sigma^2 t^2)}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Special case of the normal density:

$$\mu = 0, \quad \sigma = 1$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad M_X(t) = e^{\frac{1}{2}t^2}$$

This is the <u>standard</u> <u>normal</u> <u>density</u>

(12)

Let X have a standard normal distribution and $Y = X^2$. Find the density of Y.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$-\infty < x < \infty$$

$$G_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= (-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$g_Y(y) = \frac{d}{dy} G_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

(13)

$$= f_X(\sqrt{y})\cdot\frac{1}{2}y^{-1/2} - f_X(-\sqrt{y})\left(-\frac{1}{2}y^{-1/2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}\frac{1}{2\sqrt{y}}$$

$$g_Y(y) = \frac{1}{\sqrt{2\pi}}y^{-\frac{1}{2}}e^{-\frac{1}{2}y} \qquad y>0$$

So $Y \sim \text{Gamma}(\alpha=\frac{1}{2}, \beta=2)$

$\sim \chi^2(k=1)$

(14)

$$P(T>t) = P(X \leq \alpha-1)$$

Where $T\sim \text{Gamma}(\alpha,\beta)$ & $X \sim \text{Poisson}(\lambda = t/\beta)$

That is, T is measuring the waiting time for the α^{th} occurrence in a Poisson process.

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Example: Suppose a pizza place gets 1 order every 3 minutes, following a Poisson process.

Assume that the driver is engaged when the 5th order arrives.

Find the probability that this happens in 10 minutes or less.

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Let T = waiting time for 5 orders

$$T \sim \text{Gamma}(\alpha = 5, \underbrace{\beta = 3}_{})$$

expected time between occurrences

$$P(T \leq 10) = 1 - P(T > 10)$$

$$= 1 - P(X \leq 4) \quad \text{where } X \sim \text{Poisson}\left(\lambda = \tfrac{t}{\beta} = \tfrac{10}{3}\right)$$

$$= 1 - .7565 \quad \text{(using poissoncdf on TI-84)}$$

$$= .2435$$

Exam 3 Tuesday Nov 18

Be able to recognize and apply the following probability distributions:

Bernoulli

Binomial

Geometric

Negative Binomial

Hypergeometric

Poisson

Exponential

Gamma

3.18 There is an interesting relationship between negative binomial and gamma random variables, which may sometimes provide a useful approximation. Let Y be a negative binomial random variable with parameters r and p, where p is the success probability. Show that as $p \to 0$, the mgf of the random variable pY converges to that of a gamma distribution with parameters r and 1.

3.24 Many "named" distributions are special cases of the more common distributions already discussed. For each of the following named distributions derive the form of the pdf, verify that it is a pdf, and calculate the mean and variance.

(a) If $X \sim$ exponential(β), then $Y = X^{1/\gamma}$ has the *Weibull*(γ, β) *distribution*, where $\gamma > 0$ is a constant.

(b) If $X \sim$ exponential(β), then $Y = (2X/\beta)^{1/2}$ has the *Rayleigh distribution*.

(c) If $X \sim$ gamma(a, b), then $Y = 1/X$ has the *inverted gamma* IG(a, b) distribution. (This distribution is useful in Bayesian estimation of variances; see Exercise 7.23.)

(d) If $X \sim$ gamma($\frac{3}{2}, \beta$), then $Y = (X/\beta)^{1/2}$ has the *Maxwell distribution*.

(e) If $X \sim$ exponential(1), then $Y = \alpha - \gamma \log X$ has the *Gumbel*(α, γ) *distribution*, where $-\infty < \alpha < \infty$ and $\gamma > 0$. (The Gumbel distribution is also known as the *extreme value distribution*.)

3.39 Consider the Cauchy family defined in Section 3.3. This family can be extended to a location-scale family yielding pdfs of the form

$$f(x|\mu, \sigma) = \frac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)}, \quad -\infty < x < \infty.$$

The mean and variance do not exist for the Cauchy distribution. So the parameters μ and σ^2 are not the mean and variance. But they do have important meaning. Show that if X is a random variable with a Cauchy distribution with parameters μ and σ, then:

(a) μ is the median of the distribution of X, that is, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$.

(b) $\mu + \sigma$ and $\mu - \sigma$ are the quartiles of the distribution of X, that is, $P(X \geq \mu + \sigma) = P(X \leq \mu - \sigma) = \frac{1}{4}$. (*Hint:* Prove this first for $\mu = 0$ and $\sigma = 1$ and then use Exercise 3.38.)

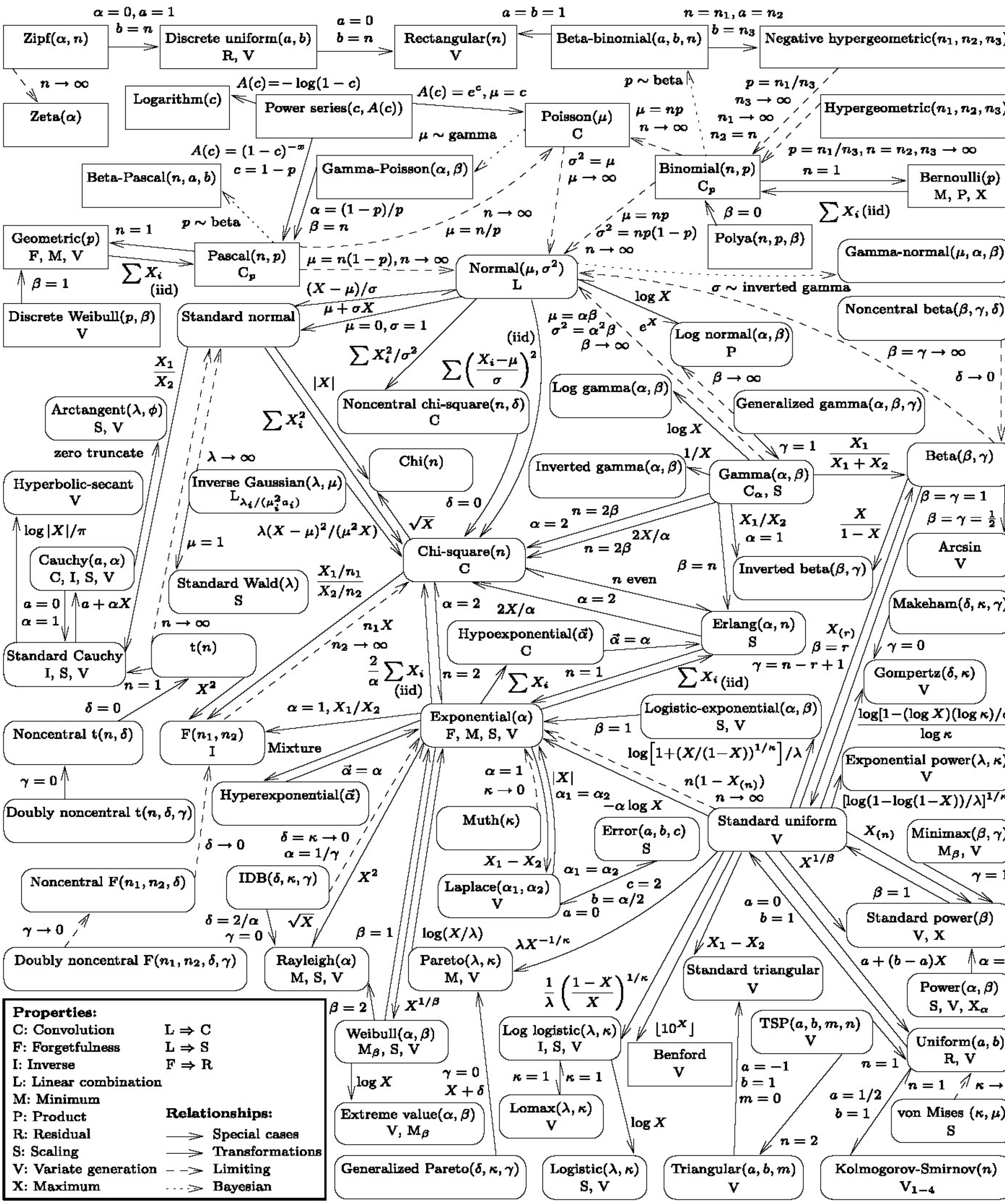

Figure 1. Univariate distribution relationships.