Stat 561

11-6-25

The Poisson Distribution

①

$$f_X(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad x = 0, 1, 2, \ldots$$

②

Start with Bino(n, p)

Let $n \to \infty$

$p \to 0$ (hold $\underbrace{np}_{\mu}$ constant)

$$p(x) = \binom{n}{x} p^x q^{n-x} \qquad p = \frac{\mu}{n}$$

$$= \frac{n!}{x!(n-x)!} \frac{\mu^x}{n^x} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

$$= \frac{\mu^x}{x!} \frac{n(n-1)\cdots(n-x+1)}{n \cdot n \cdots n} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

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$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{\mu^x}{x!} 1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

$$= \frac{\mu^x}{x!} e^{-\mu}$$

Let $y = \left(1-\frac{\mu}{n}\right)^n$

$$\ln y = n \ln\left(1-\frac{\mu}{n}\right)$$

$$= \frac{\ln\left(1-\frac{\mu}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\frac{1}{(1-\mu/n)}\cdot\frac{\mu}{\cancel{n^2}}}{-\frac{1}{\cancel{n^2}}}$$

$$= -\mu$$

$$\therefore \lim_{n\to\infty} y = e^{-\mu}$$

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Bernoulli

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Hypergeom $\xrightarrow[\substack{N\to\infty \\ \frac{M}{N}\text{ const.}}]{}$ Bino $\xrightarrow[\substack{n\to\infty \\ np\text{ const.}}]{}$ Poisson

Neg Hyper $\longrightarrow$ Neg Bino $\longrightarrow$? $\Big\}$ Not in our book

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geometric

Numerical example: Population of 10,000 people

Take a sample of 1000, without replacement

In the population, 500 have blue eyes.

Find the prob. of getting exactly 45 people in the sample with blue eyes.

Hyper$(N = 10000, M = 500, K = 1000)$

$$P(X=45) = \frac{\binom{500}{45}\binom{9500}{955}}{\binom{10000}{1000}} = .047136$$

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Do the Binomial approximation

Bino$(n = 1000, p = .05)$

$$P(X=45) = \binom{1000}{45}(.05)^{45}(.95)^{955} = .046281$$

And the Poisson approximation

Poisson$(\mu = 50)$

$$P(X=45) = \frac{50^{45}}{45!}e^{-50} = .045826$$

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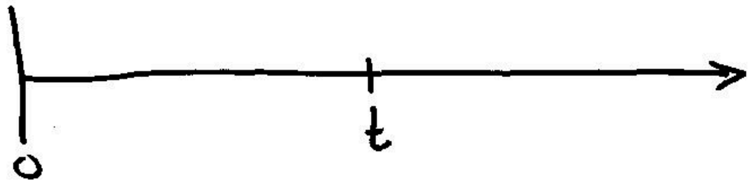
Suppose we have a Poisson process



Let T be the waiting time until the r^{th} occurrence.

Find $P(T > t) = P\left(\begin{array}{c}\text{there were at most } r-1 \text{ occurrences} \\ \text{in } (0,t)\end{array}\right)$

$$= P(X \le r-1),$$

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where X has a Poisson distribution

with $\mu = \lambda t$

λ: expected # occurrences per unit time

Special case of $r=1$:

T is the waiting time until the 1^{st} occurrence

$$P(T > t) = P(X \le 0) = P(X=0)$$

$$= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

(9)

$$F_T(t) = P(T \le t) = 1 - P(T > t)$$
$$= 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$

This is the <u>exponential</u> density function

Define the <u>gamma function</u>:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

(10)

Properties:

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = -e^{-t}\Big|_0^\infty$$
$$= 0 - (-1)$$
$$= 1$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$
$$= -t^{\alpha-1} e^{-t}\Big|_0^\infty + \int_0^\infty (\alpha-1) t^{\alpha-2} e^{-t}\, dt$$

Let $u = t^{\alpha-1}$ $\quad dv = e^{-t} dt$

$du = (\alpha-1) t^{\alpha-2} dt$ $\quad v = -e^{-t}$

$$= 0 + (\alpha-1)\int_0^\infty t^{\alpha-2}e^{-t}\,dt$$

by L'H,
for $\alpha \geq 1$

$$\therefore \Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

So $\Gamma(1) = 1$

$\Gamma(2) = 1\cdot\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2$

$\Gamma(4) = 3\Gamma(3) = 6$

Pattern:

$$\Gamma(n) = (n-1)!$$

Defn: $$f_T(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\,t^{\alpha-1}e^{-t/\beta},$$

$t > 0 \quad (\beta > 0)$

is the pdf for the Gamma distribution

Check: $$1 \overset{?}{=} \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}\,t^{\alpha-1}e^{-t/\beta}\,dt$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty (\beta u)^{\alpha-1}e^{-u}\,\beta\,du$$

$u = t/\beta$

$du = \frac{1}{\beta}\,dt$

(13)

$$= \frac{1}{\Gamma(\alpha)} \underbrace{\int_0^\infty u^{\alpha-1} e^{-u}\, du}_{\Gamma(\alpha)} = 1 \quad ✓$$

Special case: $\alpha = 1$

$$f_T(t) = \frac{1}{\Gamma(1)\beta^1} t^0 e^{-t/\beta}$$

$$= \frac{1}{\beta} e^{-t/\beta} \qquad t > 0$$

This is the exponential density with $\lambda = \frac{1}{\beta}$

Find the m.g.f.: (14)

$$M_X(t) = E[e^{tX}]$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)}\, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{1/\left(\frac{1}{\beta}-t\right)}}\, dx$$

(15)

$$= \frac{1}{\beta^{\alpha}}\left[\frac{1}{(\frac{1}{\beta}-t)}\right]^{\alpha} \underbrace{\int_0^{\infty} \frac{1}{\Gamma(\alpha)\left[\frac{1}{(\frac{1}{\beta}-t)}\right]^{\alpha}}\, x^{\alpha-1} e^{\frac{-x}{1/(\frac{1}{\beta}-t)}}\,dx}_{1}$$

$$= \frac{1}{(1-\beta t)^{\alpha}} = (1-\beta t)^{-\alpha}$$

$$M_X'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$M_X''(t) = \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)$$
$$= \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$$

(16)

$$\mu = E[X] = M_X'(0) = \alpha\beta$$

$$E[X^2] = M_X''(0) = \alpha(\alpha+1)\beta^2$$

$$\sigma^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$