Stat 561

11-4-25

The Poisson process

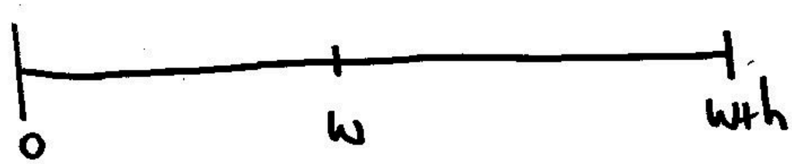
Let $f(x,h)$ be the probability of seeing exactly x occurrences of a particular type in a time interval of length h.

① $f(1,h) = \lambda h + o(h)$

[Note: a function $g(h)$ is $o(h)$ if $\lim_{h \to 0} \frac{g(h)}{h} = 0$]

② $\sum_{x=2}^{\infty} f(x,h) = o(h)$ ②

③ occurrences in disjoint time intervals are independent



$f(0,w+h) = f(0,w)\cdot f(0,h)$ by ③

Also,

$$f(0,h) = 1-\left[f(1,h) + \sum_{x=2}^{\infty} f(x,h)\right]$$

$$= 1-\left[\underbrace{\lambda h + o(h)}_{①} + \underbrace{o(h)}_{②}\right]$$

$$= 1-\lambda h + o(h)$$

③

So $$f(0,w+h) = f(0,w)\left[1-\lambda h + o(h)\right]$$

$$= f(0,w) + f(0,w)\left[-\lambda h + o(h)\right]$$

④

$$\lim_{h\to 0} \frac{f(0,w+h)-f(0,w)}{h} = \lim_{h\to 0} \frac{f(0,w)\left[-\lambda h + o(h)\right]}{h}$$

$$\frac{\partial f(0,w)}{\partial w} = -\lambda f(0,w)$$

$$\int \frac{\partial f(0,w)}{f(0,w)} = \int -\lambda \, \partial w$$

$$\ln f(0,w) = -\lambda w + c$$

$$f(0,w) = e^{-\lambda w + c}$$

5

$$f(0,0) = 1 \Rightarrow$$

$$1 = e^{-\lambda \cdot 0 + c}$$

$$\Rightarrow c = 0$$

So $f(0,w) = e^{-\lambda w}$

Suppose $x = 1$

$$f(1, w+h) = f(1,w) \cdot f(0,h) + f(0,w) \cdot f(1,h)$$

6

Generalize this for $x \geq 1$:

$$\begin{aligned} f(x, w+h) = {} & P[x \text{ in } w \cap 0 \text{ in } h] \\ & + \\ & P[x-1 \text{ in } w \cap 1 \text{ in } h] \\ & + \\ & \vdots \\ & P[0 \text{ in } w \cap x \text{ in } h] \end{aligned}$$

$$= f(x,w)f(0,h) + f(x-1,w)f(1,h) + \sum_{i=2}^{x} f(x-i,w)f(i,h)$$

(7)

$$= f(x,w)\left[1-\lambda h+o(h)\right]$$
$$+ f(x-1,w)\left[\lambda h+o(h)\right]$$
$$+ o(h)$$

$$\frac{f(x,w+h)-f(x,w)}{h} = \frac{f(x,w)\left[-\lambda h+o(h)\right] + f(x-1,w)\left[\lambda h+o(h)\right] + o(h)}{h}$$

(8)

Take $\lim_{h\to 0}$

$$\frac{\partial f(x,w)}{\partial w} = -\lambda f(x,w) + \lambda f(x-1,w)$$

for $x \geq 1$

(9)

$x=1$ case:

$$\frac{\partial f(1,w)}{\partial w} = -\lambda f(1,w) + \lambda \underbrace{f(0,w)}_{e^{-\lambda w}}$$

$$e^{\lambda w}\frac{\partial f(1,w)}{\partial w} = -\lambda e^{+\lambda w} f(1,w) + \lambda$$

$$e^{\lambda w}\frac{\partial f(1,w)}{\partial w} + \lambda e^{\lambda w} f(1,w) = \lambda$$

(10)

$$\frac{d}{dw}\left[e^{\lambda w} f(1,w)\right] = \lambda$$

$$e^{\lambda w} f(1,w) = \lambda w + c$$

$$f(1,w) = \lambda w e^{-\lambda w} + c e^{-\lambda w}$$

Use $f(1,0) = 0$

$$\therefore \underbrace{f(1,0)}_{0} = 0 + c \qquad \underline{c=0}$$

So $\underline{f(1,w) = \lambda w e^{-\lambda w}}$

(11)

$k=2$ case:

$$\frac{\partial f(2,w)}{\partial w} = -\lambda f(2,w) + \lambda \underbrace{f(1,w)}_{\lambda w e^{-\lambda w}}$$

$$e^{\lambda w}\frac{\partial f(2,w)}{\partial w} + \lambda e^{\lambda w} f(2,w) = \lambda^2 w$$

$$\frac{d}{dw}\left[e^{\lambda w} f(2,w)\right] = \lambda^2 w$$

$$e^{\lambda w} f(2,w) = \lambda^2 \frac{w^2}{2} + c$$

$$f(2,w) = \frac{\lambda^2 w^2}{2} e^{-\lambda w} + c e^{-\lambda w}$$

(12)

Use $f(2,0) = 0$

$$0 = f(2,0) = 0 + c \qquad \therefore c = 0$$

$$f(2,w) = \frac{\lambda^2 w^2}{2} e^{-\lambda w}$$

Summary:

$$f(0,w) = e^{-\lambda w}$$

$$f(1,w) = \lambda w e^{-\lambda w}$$

$$f(2,w) = \frac{(\lambda w)^2}{2} e^{-\lambda w}$$

$$\vdots$$

$$f(k,w) = \frac{(\lambda w)^k}{k!} e^{-\lambda w}$$

You usually see this as (13)

$$f_X(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

or $$f_X(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad x = 0, 1, 2, \ldots$$

This is the Poisson Distribution

Check $$1 \stackrel{?}{=} \sum_{x=0}^{\infty} \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \underbrace{\left(1 + \mu + \frac{\mu^2}{2!} + \ldots\right)}_{e^{\mu}}$$

$$= 1 \quad \checkmark$$

(14)

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x e^{-\mu}}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$

$$= e^{-\mu} e^{\mu e^t} = e^{\mu(e^t - 1)}$$

$$M_X'(t) = e^{\mu(e^t - 1)} \mu e^t$$

15

$$M_X''(t) = \mu\left[e^{\mu(e^t-1)}e^t + e^t e^{\mu(e^t-1)}\mu e^t\right]$$

$$E[X] = M_X'(0) = \mu$$

$$E[X^2] = M_X''(0) = \mu[1+\mu] = \mu + \mu^2$$

$$\sigma^2 = \mu + \mu^2 - \mu^2 = \mu$$

Stat 561 HW 5

3.2 A manufacturer receives a lot of 100 parts from a vendor. The lot will be unacceptable if more than five of the parts are defective. The manufacturer is going to select randomly K parts from the lot for inspection and the lot will be accepted if no defective parts are found in the sample.

(a) How large does K have to be to ensure that the probability that the manufacturer accepts an unacceptable lot is less than .10?

(b) Suppose the manufacturer decides to accept the lot if there is at most one defective in the sample. How large does K have to be to ensure that the probability that the manufacturer accepts an unacceptable lot is less than .10?

3.7 Let the number of chocolate chips in a certain type of cookie have a Poisson distribution. We want the probability that a randomly chosen cookie has at least two chocolate chips to be greater than .99. Find the smallest value of the mean of the distribution that ensures this probability.

3.13 A *truncated* discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range $0, 1, 2, \ldots$ and the 0 class cannot be observed (as is usually the case), the 0-*truncated* random variable X_T has pmf

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, \ldots.$$

Find the pmf, mean, and variance of the 0-truncated random variable starting from

(a) $X \sim \text{Poisson}(\lambda)$.

(b) $X \sim$ negative binomial(r, p), as in (3.2.10).

3.17 Establish a formula similar to (3.3.18) for the gamma distribution. If $X \sim \text{gamma}(\alpha, \beta)$, then for any positive constant ν,

$$\mathrm{E}X^{\nu} = \frac{\beta^{\nu}\Gamma(\nu + \alpha)}{\Gamma(\alpha)}.$$