The geometric distribution

Stat 521
10-28-25

(1)

Run a sequence of independent trials,
2 possible outcomes on each,
Constant success probability p

$\Bigg\}$ Bernoulli trials

X = trial on which 1^{st} success occurs

x	$p(x)$
1	p
2	qp
3	$q^2 p$
$\vdots$	

$$\boxed{p_X(x) = pq^{x-1} \quad x = 1,2,3,\ldots}$$

Check: $\displaystyle\sum_{x=1}^{\infty} pq^{x-1} = p[1+q+q^2+q^3+\ldots]$

$= p\,\dfrac{1}{1-q} = 1$

(2)

$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} p q^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} e^{t(x-1)} q^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t[1 + qe^t + (qe^t)^2 + \ldots]$$

$$= \frac{pe^t}{1-qe^t} \quad (\text{provided } qe^t < 1)$$

$$M_X(0) = \frac{p}{1-q} = 1 \checkmark$$

$$M_X'(t) = \frac{(1-qe^t)(pe^t) - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t}{(1-qe^t)^2} \qquad M_X'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$M_X''(t) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$M_X''(0) = \frac{(1-q)^2 p - p\,2(1-q)(-q)}{(1-q)^4}$$

$$= \frac{p^3 + 2p^2 q}{p^4}$$

$$= \frac{p+2q}{p^2} = E[X^2]$$

$$\sigma^2 = \frac{p+2q}{p^2} - \frac{1}{p^2} = \frac{1-q+2q-1}{p^2} = \frac{q}{p^2}$$

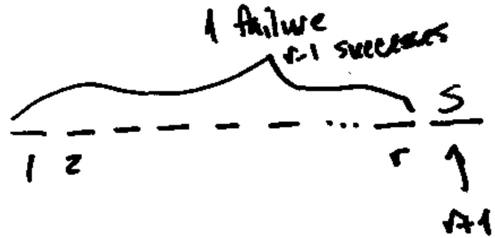
Negative Binomial Distribution (Pascal Distribution)

Run a sequence of Bernoulli trials

$X =$ trial on which the r^{th} success occurs

⑤

x	$p(x)$
r	p^r
$r+1$	$r\,p^{r-1}qp$
$r+2$	$\binom{r+1}{2}p^{r-1}q^2p$
$r+3$	$\binom{r+2}{3}p^{r-1}q^3p$
$\vdots$	



$$p_X(x) = \binom{x-1}{x-r}p^rq^{x-r}$$

$$\boxed{p_X(x) = \binom{x-1}{r-1}p^rq^{x-r}} \quad x=r, r+1, \ldots$$

Since $\binom{n}{a} = \binom{n}{n-a}$

⑥

$$M_X(t) = E[e^{tX}] = \sum_{x=r}^{\infty} e^{tx}\binom{x-1}{r-1}p^rq^{x-r}$$

$$= \sum_{x=r}^{\infty}\binom{x-1}{r-1}p^r\frac{e^{tx}q^x}{q^r}$$

$$= \frac{(qe^t)^r}{(1-qe^t)^r}\frac{p^r}{q^r}\underbrace{\sum_{x=r}^{\infty}\binom{x-1}{r-1}(1-qe^t)^r\frac{(qe^t)^x}{(qe^t)^r}}_{1}$$

$$M_X(t) = \left[\frac{pe^t}{1-qe^t}\right]^r$$

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Note: $X = X_1 + X_2 + \dots + X_r$

X_1: trial on which 1st success occurred

X_2: # of additional trials for 2nd success

Each $X_i \sim \text{Geom}(p)$

And they are indep.

So $E[X] = r\,E[X_i] = \frac{r}{p}$

$V[X] = r\,V[X_i] = \frac{rq}{p^2}$

(8)

Hypergeometric Distribution

Take a random sample of K objects, without replacement, from a population of N objects. In the population, M items are of a particular type.

X counts the number of items of that type in the sample.

$$p_X(x) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$$

$x \geq 0$

$x \leq M$, $x \leq K$

$K - x \leq N - M$

$x \geq K - N + M$

So $x \leq \min(M, K)$

$x \geq \max(0, K - N + M)$

There is no closed form for $M_X(t)$ for the hypergeometric distribution.

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$$p_X(x) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}},$$

$$\max(0, K-N+M) \le x \le \min(K, M)$$

Find $E[X] = \sum_{x=a}^{b} x \frac{\frac{M!}{x!\,(M-x)!} \cdot \frac{(N-M)!}{(K-x)!\,(N-M-K+x)!}}{\frac{N!}{K!\,(N-K)!}}$

Case 1: $a=0$

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Note that the contribution to the sum is 0 when $x=0$

$$E[X] = \sum_{x=1}^{b} \frac{\frac{M!}{(x-1)!\,(M-x)!} \cdot \frac{(N-M)!}{(K-x)!\,(N-M-K+x)!}}{\frac{N!}{K!\,(N-K)!}}$$

$\boxed{\text{Let } y = x-1}$

$$= \sum_{y=0}^{b-1} \frac{\frac{M!}{y!\,(M-y-1)!} \cdot \frac{(N-M)!}{(K-y-1)!\,(N-M-K+y+1)!}}{\frac{N!}{K!\,(N-K)!}}$$

(11)

$$\boxed{\begin{aligned} \text{Let } m &= M-1 \\ k &= K-1 \\ n &= N-1 \end{aligned}} = \sum_{y=0}^{\min(k,m)} \frac{\dfrac{(m+1)!}{y!\,(m-y)!} \; \dfrac{(n-m)!}{(k-y)!\,(n-m-k+y)!}}{\dfrac{(n+1)!}{(k+1)!\,(n-k)!}}$$

$$= \frac{(m+1)(k+1)}{n+1} \underbrace{\sum_{y=0}^{\min(k,m)} \frac{\binom{m}{y}\binom{n-m}{k-y}}{\binom{n}{k}}}_{1}$$

(12)

$$E[X] = \frac{MK}{N} = K \cdot \frac{M}{N}$$

(Note that this is similar to $\mu = np$ for binomial)

Without going through the derivation,

$$V[X] = K \frac{M}{N}\left(1 - \frac{M}{N}\right) \underbrace{\left(\frac{N-K}{N-1}\right)}_{f_{pc}}$$

"finite population correction"

See what happens to the hypergeometric

if we let $N \to \infty$

and hold $\frac{M}{N}$ constant

Let $p = \frac{M}{N}$

$$p_X(x) = \frac{\binom{Np}{x}\binom{N-Np}{K-x}}{\binom{N}{K}}$$

$$= \frac{(Np)!}{x!\,(Np-x)!} \cdot \frac{(N-Np)!}{(K-x)!\,(N-Np-K+x)!} \cdot \frac{K!\,(N-K)!}{N!}$$

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$$= \frac{K!}{x!\,(K-x)!} \frac{\overbrace{(Np)(Np-1)\cdots(Np-x+1)}^{x \text{ terms}}\overbrace{(N-Np)\cdots(N-Np-K+x+1)}^{K-x \text{ terms}}}{\underbrace{N(N-1)\cdots(N-K+1)}_{K \text{ terms}}}$$

$$= \binom{K}{x} \frac{p\left(p-\frac{1}{N}\right)\cdots\left(p-\frac{x-1}{N}\right)(q)\left(q-\frac{1}{N}\right)\cdots\left(q-\frac{K-x-1}{N}\right)}{1\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{K-1}{N}\right)}$$

Take $\lim_{N\to\infty} p_X(x) = \binom{K}{x} p^x q^{K-x}$