

Notation note from last time:

Stat 561

10-21-28

μ'_n is the n^{th} moment, $E[X^n]$

①

Also, μ_n is the n^{th} central moment, $E[(X-\mu)^n]$

$$E[X] = \mu = \mu'_1$$

Note: $\mu_1 = E[X-\mu] = E[X] - \mu = 0$

$$E[X^2] = \mu'_2$$

Note $\mu_2 = E[(X-\mu)^2]$
 $= V[X] = \sigma^2$
 $= E[X^2] - (E[X])^2$
 $= \mu'_2 - (\mu'_1)^2$

Properties of $V[X]$

②

$$\begin{aligned} V[k] &= E[k^2] - (E[k])^2 \\ &= k^2 - k^2 = 0 \end{aligned}$$

$$\begin{aligned} V[aX] &= E[a^2 X^2] - (E[aX])^2 \\ &= E[a^2 X^2] - (a E[X])^2 \\ &= a^2 E[X^2] - a^2 (E[X])^2 \\ &= a^2 [E[X^2] - (E[X])^2] \\ &= a^2 V[X] \end{aligned}$$

$$\begin{aligned}
 & V[g(X) + h(X)] \quad (3) \\
 &= E[(g(X) + h(X))^2] - (E[g(X) + h(X)])^2 \\
 &= E[(g(X))^2 + (h(X))^2 + 2g(X)h(X)] \\
 &\quad - (E[g(X)] + E[h(X)])^2 \\
 &= \underline{E[(g(X))^2]} + \underline{E[(h(X))^2]} + 2 \underline{E[g(X)h(X)]} \\
 &\quad - \left\{ \underline{(E[g(X)])^2} + \underline{(E[h(X)])^2} + 2 \underline{E[g(X)]E[h(X)]} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= V[g(X)] + V[h(X)] \quad (4) \\
 &\quad + 2 \left[\underbrace{E[g(X)h(X)] - E[g(X)]E[h(X)]}_{\text{This is the covariance of } g(X) \text{ with } h(X)} \right]
 \end{aligned}$$

The variance is a quadratic operator

$$\begin{aligned}
 V[aX + b] &= V[aX] + V[b] + \\
 &\quad 2[E[abX] - E[aX]E[b]] \\
 &= a^2 V[X] + 0 + 2[\cancel{abE[X]} - aE[X]b]
 \end{aligned}$$

$$\text{Defn: } M_X(t) = E[e^{tX}] \quad (5)$$

This is called the moment-generating function for the r.v. X .

$$\text{Properties: } M_X(0) = E[e^0] = 1$$

$$\begin{aligned} \frac{d}{dt} M_X(t) &= \frac{d}{dt} E[e^{tX}] \\ &= E\left[\frac{d}{dt} e^{tX}\right] \quad \text{why?} \end{aligned}$$

$$\begin{aligned} \text{Because: } \frac{d}{dt} \int_a^b f(x,t) dx & \quad (6) \\ &= \lim_{h \rightarrow 0} \frac{\int_a^b f(x,t+h) dx - \int_a^b f(x,t) dx}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_a^b [f(x,t+h) - f(x,t)] dx}{h} \\ &= \lim_{h \rightarrow 0} \int_a^b \frac{f(x,t+h) - f(x,t)}{h} dx \end{aligned}$$

$$\begin{aligned}
 &= \int_a^b \lim_{h \rightarrow 0} \frac{f(x, t+h) - f(x, t)}{h} dx \quad (7) \\
 &\quad \text{by Leibniz' Theorem} \\
 &= \int_a^b \frac{\partial}{\partial t} f(x, t) dx
 \end{aligned}$$

$$\begin{aligned}
 \text{So } \frac{d}{dt} M_X(t) &= \frac{d}{dt} E[e^{tX}] = E\left[\frac{d}{dt} e^{tX}\right] \\
 &= E[X e^{tX}] \\
 M'_X(0) &= E[X]
 \end{aligned}$$

$$\begin{aligned}
 M''_X(t) &= E[X^2 e^{tX}] \quad (8) \\
 M''_X(0) &= E[X^2]
 \end{aligned}$$

Example: $f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, x > 0$

$$\begin{aligned}
 M_X(t) &= E[e^{tX}] = \int_0^{\infty} e^{tx} \frac{1}{\lambda} e^{-x/\lambda} dx \\
 &= \frac{1}{\lambda} \int_0^{\infty} e^{x(t - \frac{1}{\lambda})} dx \\
 &= \frac{1}{\lambda} \frac{e^{x(t - \frac{1}{\lambda})}}{t - \frac{1}{\lambda}} \bigg|_{x=0}^{\infty}
 \end{aligned}$$

$$= \frac{1}{\lambda} \left[0 - \frac{1}{t - \frac{1}{\lambda}} \right] \text{ provided } t < \frac{1}{\lambda} \quad (9)$$

$$= \frac{-1}{t\lambda - 1}$$

$$\boxed{M_x(t) = \frac{1}{1 - \lambda t}} = (1 - \lambda t)^{-1} \quad [M_x(0) = 1 \checkmark]$$

$$M_x'(t) = -(1 - \lambda t)^{-2}(-\lambda) = \lambda(1 - \lambda t)^{-2}$$

$$M_x''(t) = \lambda(-2)(1 - \lambda t)^{-3}(-\lambda) = 2\lambda^2(1 - \lambda t)^{-3}$$

$$M_x'(0) = \lambda = E[X] \quad M_x''(0) = 2\lambda^2 = E[X^2]$$

$$\sigma^2 = 2\lambda^2 - \lambda^2 = \lambda^2$$

Example: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, \dots, n \quad (10)$

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^n e^{tk} \binom{n}{k} p^k q^{n-k}$$

($q = 1 - p$)

$$= \sum_{k=0}^n \binom{n}{k} (pe^{tX})^k q^{n-k}$$

$$= (pe^t + q)^n \quad \text{by binomial theorem}$$

Check $M_X(0) = (p+q)^n = 1 \checkmark$

$$M_X'(t) = n(p e^t + q)^{n-1} p e^t$$

$$= n p e^t (p e^t + q)^{n-1}$$

(11)

$$M_X''(t) = n p \left[e^t (n-1) (p e^t + q)^{n-2} p e^t + e^t (p e^t + q)^{n-1} \right]$$

$$M_X'(0) = n p = E[X]$$

$$M_X''(0) = n p [(n-1)p + 1] = n(n-1)p^2 + n p$$

$$= E[X^2]$$

$$\sigma^2 = n(n-1)p^2 + n p - n^2 p^2$$

$$= \cancel{n^2 p^2} - n p^2 + n p - \cancel{n^2 p^2} = n p (1-p) = n p q$$

Defn: The standard deviation of a r.v. X

$$\text{is } \sigma = \sqrt{V[X]}$$

(12)

Defn: The coefficient of variation of a r.v. X

$$\text{is } CV = \frac{\sigma}{\mu}$$

Chebyshev's Theorem: Let $g(X)$ be a

non-negative function of the r.v. X

and let r be a positive constant

$$\text{Then } P[g(X) \geq r] \leq \frac{E[g(X)]}{r} \quad (13)$$

Proof: (continuous case)

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$\text{Let } A = \{x \mid g(x) \geq r\}$$

$$\text{Then } E[g(X)] = \int_A g(x) f_X(x) dx + \int_{A^c} g(x) f_X(x) dx$$

$$\geq \int_A g(x) f_X(x) dx$$

$$\geq \int_A r f_X(x) dx$$

(14)

$$= r \int_A f_X(x) dx$$

$$= r P(X \in A)$$

$$E[g(X)] \geq r P(g(X) \geq r)$$

$$P(g(X) \geq r) \leq \frac{E[g(X)]}{r}$$