

Another transformation example:

Stat 561
10-16-25

①

Let X have cdf $F_X(x)$ and

assume that F is continuous, and
monotone increasing

Define $Y = F(X)$ $\uparrow$

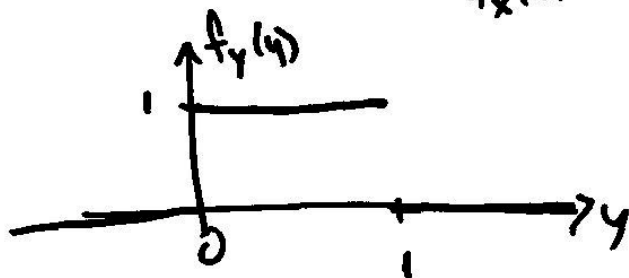
Find the pdf for Y .

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f_X(x)}$$

②

$$\text{So } f_Y(y) = f_X(x) \frac{1}{f_X(x)} = 1 \text{ for } 0 \leq y \leq 1$$



Uniform density

Expectation operator

(3)

Defn: If X is a discrete r.v. and
 $g(x)$ is any function of X ,

$$\text{then } E[g(X)] = \sum_{\text{all } x} g(x) p_x(x)$$

$$\left(\text{provided that } \sum_{\text{all } x} |g(x)| p_x(x) < \infty \right)$$

If X is a continuous r.v.

(4)

And $g(x)$ is any function of X ,

$$\text{then } E[g(X)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

$$\left(\text{provided that } \int_{-\infty}^{\infty} |g(x)| f_x(x) dx < \infty \right)$$

Special Cases: $E[X] = \begin{cases} \sum_{\text{all } x} x p_x(x) \\ \int_{-\infty}^{\infty} x f_x(x) dx \end{cases}$

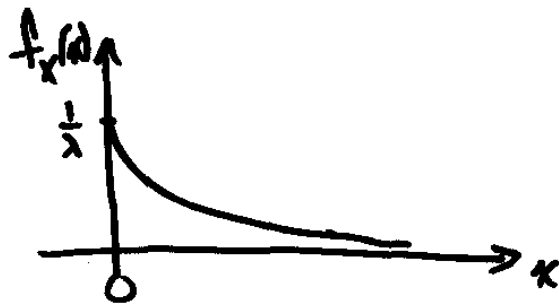
$$E[X^n] = \begin{cases} \sum_{\text{all } x} x^n p_X(x) \\ \int_{-\infty}^{\infty} x^n f_X(x) dx \end{cases} \quad (5)$$

Defn: $E[X^n]$ is the n^{th} moment of the random variable X .

Notation: $\mu'_n = E[X^n]$
 (So $\mu'_1 = E[X] = \mu$)

Example: Let $f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}$, $x > 0$ (6)

Find $E[X]$



$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x \frac{1}{\lambda} e^{-x/\lambda} dx \quad \left| \begin{array}{l} \text{let } u = x \\ dv = \frac{1}{\lambda} e^{-x/\lambda} dx \end{array} \right. \\ &= \int_0^{\infty} u dv = uv \Big|_0^{\infty} - \int_0^{\infty} v du \end{aligned}$$

$$\begin{aligned}
&= -\lambda e^{-x/\lambda} \Big|_0^\infty + \int_0^\infty e^{-x/\lambda} dx \quad \left| \begin{array}{l} du = dx \\ v = -e^{-x/\lambda} \end{array} \right. \textcircled{7} \\
&= \frac{-\lambda}{e^{x/\lambda}} \Big|_0^\infty - \lambda e^{-x/\lambda} \Big|_0^\infty \\
&= \underset{\substack{\uparrow \\ \text{by L'Hôpital}}}{0} - (0) - [0 - \lambda] = \lambda
\end{aligned}$$

Example: let $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ \textcircled{8}
 $x = 0, 1, 2, \dots, n$

Binomial theorem:

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$$

Note: $\sum_{\text{all } x} p_X(x) = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x}$
 $= [p + (1-p)]^n = 1$

Find $E[X]$

(9)

$$\mu = \sum_{\text{all } x} x p_x(x) = \sum_{x=0}^n x \binom{n}{x} p^x q^{n-x}$$

$(q = 1 - p)$

$$= \sum_{x=1}^n x \binom{n}{x} p^x q^{n-x}$$

Let $y = x - 1$
 $x = y + 1$

$$= \sum_{y=0}^{n-1} (y+1) \binom{n}{y+1} p^{y+1} q^{n-(y+1)}$$

$$= \sum_{y=0}^{n-1} (y+1) \frac{n!}{(y+1)!(n-y-1)!} p^{y+1} q^{n-y-1}$$

$$= \sum_{y=0}^{n-1} \frac{n!}{y!(n-y-1)!} p^{y+1} q^{n-y-1} \quad (10)$$

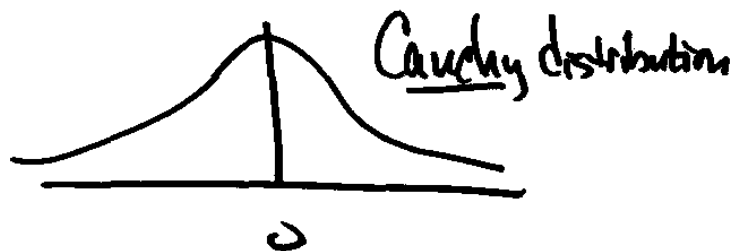
Let $m = n - 1$
 $n = m + 1$

$$= \sum_{y=0}^m \frac{(m+1)!}{y!(m-y)!} p^{y+1} q^{m-y}$$

$$= (m+1) p \underbrace{\sum_{y=0}^m \frac{m!}{y!(m-y)!} p^y q^{m-y}}_1$$

$$= np$$

Example: $f_x(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad -\infty < x < \infty$ (11)



Find $E[X]$. Check the extra condition

$$\int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2} dx < \infty ?$$

$$= 2 \int_0^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx \quad \begin{array}{l} \text{let } u = 1+x^2 \\ du = 2x dx \end{array}$$

$$= \frac{1}{\pi} \int_1^{\infty} \frac{du}{u} = \frac{1}{\pi} \ln u \Big|_1^{\infty} = \infty \quad (12)$$

$\therefore E[X]$ will be undefined

Properties of the expectation operator:

$$\textcircled{1} E(k) = \int_{-\infty}^{\infty} k f_x(x) dx = k \int_{-\infty}^{\infty} f_x(x) dx$$

$$= k$$

(similar in discrete case)

$$\begin{aligned} \textcircled{2} E[kg(x)] &= \int_{-\infty}^{\infty} kg(x)f_x(x)dx \\ &= k \int_{-\infty}^{\infty} g(x)f_x(x)dx = k E[g(x)] \end{aligned} \quad \textcircled{13}$$

$$\begin{aligned} \textcircled{3} E[g(x) + h(x)] &= \int_{-\infty}^{\infty} [g(x) + h(x)]f_x(x)dx \\ &= \int_{-\infty}^{\infty} g(x)f_x(x)dx + \int_{-\infty}^{\infty} h(x)f_x(x)dx \\ &= E[g(x)] + E[h(x)] \end{aligned}$$

Note: operators with properties $\textcircled{2}$ and $\textcircled{3}$ are called linear operators

$\textcircled{14}$

Defn: The variance of a random variable

$$\begin{aligned} \text{is } \sigma^2 &= V[X] = \text{Var}[X] \\ &= E[(X - \mu)^2] \end{aligned}$$

$$\begin{aligned} \text{Alternate form: } \sigma^2 &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \end{aligned}$$

(15)

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$\begin{aligned}\sigma^2 &= E[X^2] - \mu^2 \\ &= E[X^2] - (E[X])^2\end{aligned}$$

2.2 In each of the following find the pdf of Y .

(a) $Y = X^2$ and $f_X(x) = 1, 0 < x < 1$

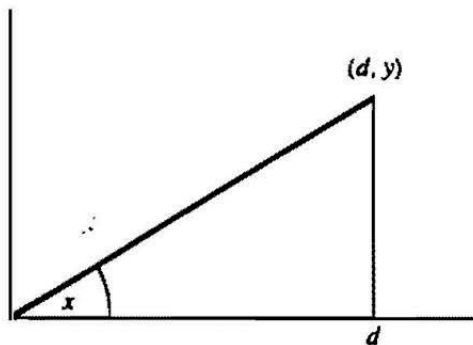
(b) $Y = -\log X$ and X has pdf

$$f_X(x) = \frac{(n+m+1)!}{n!m!} x^n (1-x)^m, \quad 0 < x < 1, \quad m, n \text{ positive integers}$$

(c) $Y = e^X$ and X has pdf

$$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}, \quad 0 < x < \infty, \quad \sigma^2 \text{ a positive constant}$$

2.12 A random right triangle can be constructed in the following manner. Let X be a random angle whose distribution is uniform on $(0, \pi/2)$. For each X , construct a triangle as pictured below. Here, Y = height of the random triangle. For a fixed constant d , find the distribution of Y and EY .



2.17 A *median* of a distribution is a value m such that $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^m f(x) dx = \int_m^{\infty} f(x) dx = \frac{1}{2}$.) Find the median of the following distributions.

(a) $f(x) = 3x^2, \quad 0 < x < 1$

(b) $f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$