Stat 561
10-9-25

①

Theorem: A function $F(x)$ is a cumulative distribution function for some random variable X if and only if

(1) $\lim_{x \to -\infty} F(x) = 0$ & $\lim_{x \to \infty} F(x) = 1$

(2) $F(x)$ is nondecreasing

(3) $F(x)$ is right-continuous

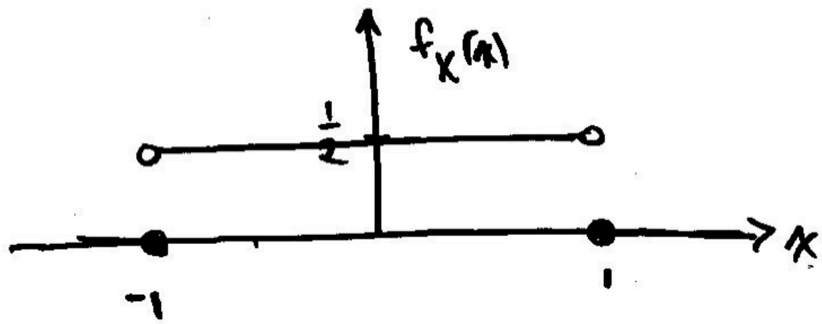
②

Defn: If there exists a function $f_X(x)$ that satisfies $\int_{-\infty}^{x} f_X(t)\,dt = F_X(x)$,

then $f_X(x)$ is the probability density function (pdf) for the random variable X.

Note: If $F_X(x)$ has a derivative $\forall x$, then $\frac{d}{dx}F_X(x)$ will be the pdf, by the Fundamental Theorem of Calculus.

<u>Example</u>: let $f_X(x) = \begin{cases} 1/2 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



Let $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x \le -1 \\ (x+1)/2 & -1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$

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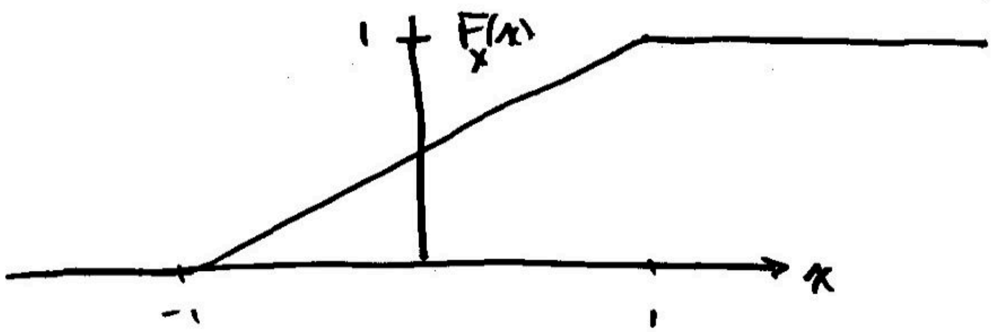

This satisfies the 3 conditions of the theorem,
so $F_X(x)$ is a valid cdf.

Note: Suppose $F_X(x)$ is continuous $\leftarrow P(X \le x)$

What happens if you try to evaluate $P(X = x)$
Consider $P(X \le x) - P(X \le x - \frac{1}{n})$ + take lim as $n \to \infty$

$$\lim_{n \to \infty} \left(F_X(x) - F(x - \tfrac{1}{n}) \right) = 0$$

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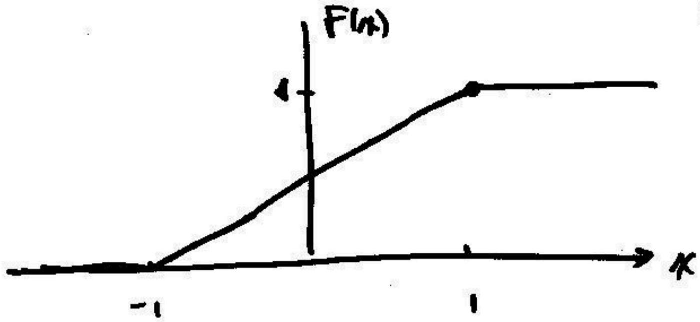
Transformations of random variables

Example : $F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2}(x+1) & -1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$



Let $Y = X^2$ Find its cdf.

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$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(X \le \sqrt{y}) - P(X < -\sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

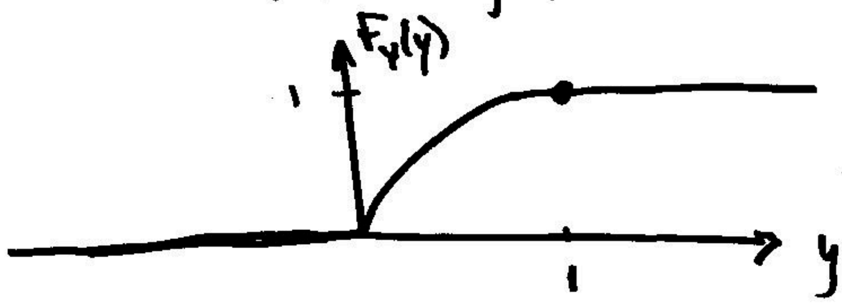
$(X \le \sqrt{y}) \cap (X < -\sqrt{y})^c$

Note: remember that our F was continuous

(7)

$$= \begin{cases} 0 & y<0 \\ \frac{1}{2}(\sqrt{y}+1) - \frac{1}{2}(-\sqrt{y}+1) & 0 \le y \le 1 \\ 1 & y>1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y<0 \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & y>1 \end{cases}$$



(8)

Find the pdf for Y

Since $F_Y(y)$ was continuous, find $f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

<u>Special cases</u>: Assume the cdf is continuous

Let $Y = g(X)$ be a tranformation of X.

Suppose $g\uparrow$ (monotone increasing)

(9)

$$F_Y(y) = P(Y \leq y)$$
$$= P(g(X) \leq y)$$
$$= P(X \leq g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

Now $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

(10)

$f_Y(y) = f_X(x) \cdot \frac{dx}{dy}$ But $y = g(x)$

so $x = g^{-1}(y)$

Suppose $g \downarrow$

$$F_Y(y) = P(Y \leq y)$$
$$= P(g(X) \leq y)$$
$$= P(X \geq g^{-1}(y))$$
$$= 1 - P(X < g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$ works since F was cont.

(11)

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right)$$

$$= -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= -f_X(x) \frac{dx}{dy}$$

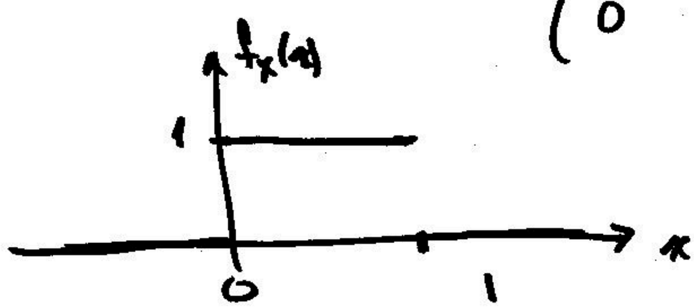
Summary: Case 1 $g \uparrow$ | Case 2 $g \downarrow$

$f_Y(y) = f_X(x) \frac{dx}{dy}$ | $f_Y(y) = -f_X(x) \frac{dx}{dy}$

$$\boxed{f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|}$$

(12)

<u>Example</u>: Let X be a continuous r.v. with pdf $f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



Let $Y = -2 \ln X$ Note: $\downarrow$

Find the pdf for Y

(13)

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= 1 \cdot \frac{1}{2}e^{-\frac{1}{2}y}$$

$y = -2\ln x$

$\ln x = -\frac{1}{2}y$

$x = e^{-\frac{1}{2}y}$

$\frac{dx}{dy} = -\frac{1}{2}e^{-\frac{1}{2}y}$

for $0 < x < 1$

$0 < e^{-\frac{1}{2}y} < 1$

$-\infty < -\frac{1}{2}y < 0$

$\infty > y > 0$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}y} & 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(14)

<u>Example</u>: $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ $\quad -\infty < x < \infty$

Let $Y = X^2$ Not monotone, so <u>no shortcut</u>

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Unfortunately, there is no closed form for $F_X(x)$

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$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y}) \right]$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \frac{-1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{\sqrt{y}} \quad \text{for} \quad -\infty < x < \infty, \; y \neq 0$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Midterm exam problem types: 16

1. Counting rules
2. Intersections, unions, disjoint, independent
3. Bayes' rule
4. A discrete distribution
5. A continuous distribution

Stat 561 HW 2

1.39 A pair of events A and B cannot be simultaneously *mutually exclusive* and *independent*. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

(a) If A and B are mutually exclusive, they cannot be independent.

(b) If A and B are independent, they cannot be mutually exclusive.

1.47 Prove that the following functions are cdfs.

(a) $\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$, $x \in (-\infty, \infty)$ (b) $(1 + e^{-x})^{-1}$, $x \in (-\infty, \infty)$

(c) $e^{-e^{-x}}$, $x \in (-\infty, \infty)$ (d) $1 - e^{-x}$, $x \in (0, \infty)$

(e) the function defined in (1.5.6)

1.54 For each of the following, determine the value of c that makes $f(x)$ a pdf.

(a) $f(x) = c \sin x$, $0 < x < \pi/2$ (b) $f(x) = ce^{-|x|}$, $-\infty < x < \infty$

1.55 An electronic device has lifetime denoted by T. The device has value $V = 5$ if it fails before time $t = 3$; otherwise, it has value $V = 2T$. Find the cdf of V, if T has pdf

$$f_T(t) = \frac{1}{1.5}e^{-t/(1.5)}, \quad t > 0.$$