Stat 561
10-2-25

Suppose we have n events $A_1, \ldots, A_n$, ①

with probabilities $p_1, p_2, \ldots, p_n$

Find a lower bound on $P(A_1 \cap \ldots \cap A_n)$.

$$P(A_1 \cap \ldots \cap A_n)^c = 1 - P(A_1 \cap \ldots \cap A_n)$$

$$\shortparallel$$

$$P(A_1^c \cup \ldots \cup A_n^c) \leq \sum_{i=1}^{n} P(A_i^c) \qquad \text{(Boole)}$$

$$= \sum_{i=1}^{n} [1 - P(A_i)] = n - \sum_{i=1}^{n} P(A_i)$$

②

$$1 - P(A_1 \cap \ldots \cap A_n) \leq n - \sum_{i=1}^{n} P(A_i)$$

$$P(A_1 \cap \ldots \cap A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

[Bonferroni inequality]

③

Example of its usage: Assume $p(A_i) = p \ \forall i$

What must p be, in order to guarantee that $P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq .95$

$$\text{Set} \quad \sum_{i=1}^{n} p_i - (n-1) = .95$$

$$np - (n-1) = .95$$

$$np = .95 + n - 1$$

$$= n - .05$$

$$p = 1 - \frac{.05}{n}$$

④

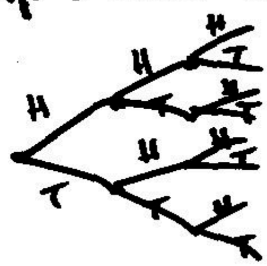
Interpretation: If you do 10 simultaneous confidence intervals, each at 99.5% confidence, you will have joint confidence of at least 95%.

⑤

Counting Rules

(1) Multiplication rule: If you perform a sequence of procedures, then the total # of possible outcomes is the product of the # of possible outcomes at each step.

Ex: Flip 3 coins in sequence. $2 \times 2 \times 2 = 8$



⑥

(2) Factorial rule: The # of ways of arranging or ordering n objects is $n!$

$\underline{n} \times \underline{n-1} \times \underline{n-2} \times \underline{\quad} \ldots \times \underline{1}$

(3) Permutation rule: Start with n objects. The # of ways of selecting and ordering r of them is

$$P_r^n = \frac{n!}{(n-r)!}$$

⑦

$$\underbrace{\underline{n} \times \underline{n-1} \times \underline{n-2} \times \underline{\quad} \cdots \times \underline{n-(r-1)}}_{r \text{ items}}$$

Note: $$\frac{n(n-1)(n-2)\cdots(n-(r-1))\cdot(n-r)(n-r-1)\cdots 1}{(n-r)(n-r-1)\cdots 1}$$

$$= \frac{n!}{(n-r)!}$$

⑧

④ Combination rule: The number of ways of selecting r objects from n objects, without regard to order, is

$$C^n_r = \binom{n}{r} = \text{"}n \text{ choose } r\text{"}$$

$$= \frac{P^n_r}{r!} = \frac{n!}{r!(n-r)!}$$

⑤ Permutations of like objects:
If there are n objects of k distinct types, then the # of ways that they can be ordered is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$, where n_i is the # of items of type i

and $n_1 + \ldots + n_k = n$

Example: How many different ways can the letters in "STATISTICS" be arranged?

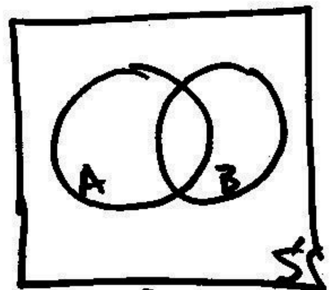
Type 1:	S	$n_1 = 3$	
2:	T	$n_2 = 3$	$n = 10$
3:	A	$n_3 = 1$	
4:	I	$n_4 = 2$	
5:	C	$n_5 = 1$	

$$\frac{10!}{3!\,3!\,1!\,2!\,1!} = \binom{10}{3\ 3\ 1\ 2\ 1} = 50{,}400$$

Conditional Probability

Defn: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$

"given"



Is this a valid probability function?

(1) $\forall A$, is $P(A|B) \geq 0$ yes

(12)

(2) $P(S|B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$ ✓

(3) Suppose $A_1 \cap A_2 = \phi$

$$P(A_1 \cup A_2 | B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$ ✓

(13)

<u>Defn</u>: The events A & B are <u>independent</u>

if $P(A \cap B) = P(A)P(B)$

Note: Suppose $P(B) > 0$ and that A and B are independent.

Then $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)}$

$= P(A)$

(14)

<u>Ex</u>: Roll 2 dice

$$S = \left\{ \begin{matrix} 11 & 12 & \cdots & 16 \\ 21 & 22 & \cdots & 26 \\ \vdots & & & \\ 61 & 62 & \cdots & 66 \end{matrix} \right\}$$

Let A be the event that the 1^{st} die $= 6$

B " " " " " 2^{nd} " $= 6$

$P(A) = \dfrac{6}{36} = \dfrac{1}{6}$ $\quad P(B) = \dfrac{6}{36} = \dfrac{1}{6}$

$P(A \cap B) = \dfrac{1}{36}$ $\quad$ Since $\dfrac{1}{36} = \dfrac{1}{6} \cdot \dfrac{1}{6}$, A and B are independent

Ex: Draw 2 cards from a deck of 52 without replacement

let A: 1^{st} card is an ace
B: 2^{nd} " " " "

$$P(A \cap B) = \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{13} \cdot \frac{1}{17} \neq \frac{1}{13} \cdot \frac{1}{13}$$

$$P(A) = \frac{4 \cdot 51}{52 \cdot 51} = \frac{4}{52} = \frac{1}{13}$$

$\therefore$ A and B are dependent

$$P(B) = \frac{4 \cdot 3 + 48 \cdot 4}{52 \cdot 51} = \frac{4}{52} = \frac{1}{13}$$

Suppose $C_1, C_2, \ldots$ is a partition of S

From last time,

$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$$

$$= \sum_{i=1}^{\infty} P(C_i)\, P(A \mid C_i) \quad \left[\begin{array}{l}\text{exclude } i \\ \text{if } P(C_i)=0\end{array}\right]$$

This is the

Law of Total Probability

Stat 561 HW 1

1.13 If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint? Explain.

1.19 If a multivariate function has continuous partial derivatives, the order in which the derivatives are calculated does not matter. Thus, for example, the function $f(x, y)$ of two variables has equal third partials

$$\frac{\partial^3}{\partial x^2 \partial y} f(x, y) = \frac{\partial^3}{\partial y \partial x^2} f(x, y).$$

(a) How many fourth partial derivatives does a function of three variables have?

(b) Prove that a function of n variables has $\binom{n+r-1}{r}$ rth partial derivatives.

1.20 My telephone rings 12 times each week, the calls being randomly distributed among the 7 days. What is the probability that I get at least one call each day?

1.27 Verify the following identities for $n \geq 2$.

(a) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

(b) $\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$

(c) $\sum_{k=1}^{n} (-1)^{k+1} k \binom{n}{k} = 0$

1.38 Prove each of the following statements. (Assume that any conditioning event has positive probability.)

(a) If $P(B) = 1$, then $P(A|B) = P(A)$ for any A.

(b) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

(c) If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

(d) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.