Stat 561
9-30-25

①

Defn: An *experiment* is a process that leads to one of several possible outcomes.

Defn: The *sample space* (S) is the set of all possible outcomes of an experiment.

Defn: An *event* is a subset of S

$\subset$: subset	A^c : complement of A
$\cap$: intersection	$\phi = \{\}$: empty set
$\cup$: union	

②

Defn: Two events A and B are *disjoint* or *mutually exclusive* if $A \cap B = \phi$

Defn. Let $\mathcal{B}$ be a subset of all possible subsets of S. Then $\mathcal{B}$ is a *Borel field* or a *σ-algebra* if

(1) $\phi \in \mathcal{B}$

(2) If $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$

(3) If $\{A_1, A_2, \dots\} \subset \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

(3)

<u>Defn</u>: Let $\mathcal{B}$ be a Borel field. Then P is a <u>probability set function</u> if

$$\left\{\begin{array}{l} (1)\ \forall\ A \in \mathcal{B},\quad P(A) \geq 0 \\ (2)\ P(S) = 1 \\ (3)\ \text{If } A_1, A_2, A_3, \ldots \text{ are pairwise disjoint,} \\ \quad \text{then } P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \end{array}\right.$$

Kolmogorov properties

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<u>Example</u>: Roll a 6-sided die

$S = \{1,2,3,4,5,6\}$

Let $\mathcal{B} = \{\phi, S, \{1,2\}, \{3,4,5,6\}\}$

This is a well-defined Borel field

Define $P_1(A) = \begin{cases} 1 & \text{if } 2 \in A \\ 0 & \text{otherwise} \end{cases}$

This is a well-defined probability function

Define $P_2(A) = \dfrac{\#\text{elements in } A}{6}$

This is also a well-defined probability function

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<u>Theorem</u>: $\forall\ A \in \mathcal{B},\ P(A^c) = 1 - P(A)$

<u>Pf</u>: $A \cap A^c = \emptyset$

$A \cup A^c = S$

$$\underset{\substack{\parallel \\ P(S) \,=\, 1}}{P(A \cup A^c)} = P(A) + P(A^c) \qquad \text{[by K-3]} \qquad \text{[by K-2]}$$

<u>Theorem</u>: $P(\emptyset) = 0$

<u>Pf</u>: $\emptyset \cap S = \emptyset$

$\emptyset \cup S = S$

$$\underset{\substack{\parallel \\ P(S) \,=\, 1}}{P(\emptyset \cup S)} = P(\emptyset) + \underbrace{P(S)}_{1}$$

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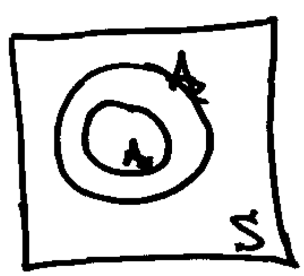
<u>Theorem</u>: Let A_1 and A_2 be elements of $\mathcal{B}$ with $A_1 \subset A_2$. Then $P(A_1) \le P(A_2)$

<u>Pf</u>: Let $A_1 \subset A_2$

Write $A_2 = A_1 \cup (A_2 \cap A_1^c)$

Also, $A_1 \cap (A_2 \cap A_1^c) = \emptyset$



So $P(A_2) = P(A_1) + \underbrace{P(A_2 \cap A_1^c)}_{\ge 0}$

$\therefore\ P(A_2) \ge P(A_1)$

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Thm: $P(A \cap B^c) = P(A) - P(A \cap B)$

Pf: $A = (A \cap B) \cup (A \cap B^c)$

$= A \cap (B \cup B^c)$ by distrib. law

$= A \cap S$

$= A$

So $P(A) = P(A \cap B) + P(A \cap B^c)$

Defn: $A \setminus B = A \cap B^c$

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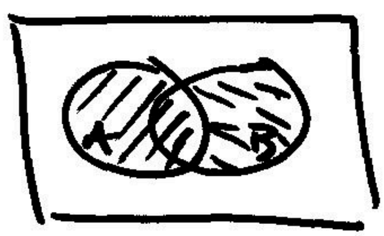
Theorem: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Pf: $A \cup B = A \cup (B \cap A^c)$

$= (A \cup B) \cap (A \cup A^c)$

$= (A \cup B) \cap S$

$= A \cup B$



$P(A \cup B) = P(A) + P(B \cap A^c)$

$= P(A) + P(B) - P(B \cap A)$

(9)

Defn: $C_1, C_2, C_3, \ldots$ form a <u>partition</u> of S if

(1) $\bigcup_{i=1}^{\infty} C_i = S$ and

(2) $C_i \cap C_j = \emptyset$ for $i \neq j$

Thm: If $C_1, C_2, \ldots$ is a partition of S, then $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$

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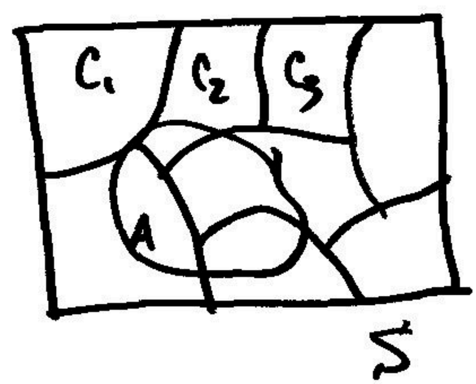
Pf:
$$A = A \cap S$$
$$= A \cap \left(\bigcup_{i=1}^{\infty} C_i \right)$$
$$= \bigcup_{i=1}^{\infty} (A \cap C_i)$$



$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$$ by Kolmogorov (3)

(11)

<u>Thm</u>: let $A_1, A_2, \ldots$ be events

Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$ [Boole's Ineq.]

<u>Pf</u>: let

$$A_1^* = A_1$$
$$A_2^* = A_2 \setminus A_1$$
$$A_3^* = A_3 \setminus (A_1 \cup A_2)$$
$$\vdots$$
$$A_i^* = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j\right)$$

Note: A_i^* and A_k^* are disjoint for $i \neq k$

(12)

Also note: $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i^*$

Then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i^*\right)$$
$$= \sum_{i=1}^{\infty} P(A_i^*)$$
$$\leq \sum_{i=1}^{\infty} P(A_i)$$ since $A_i^* \subset A_i$