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① 5-31

Recall

$$M_x(t) = E[e^{tX}]$$

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Let $X_1, X_2, \dots, X_n$ be independent random variables, with unknown distribution, with mean $\mu$ and standard deviation $\sigma$.

Let $Z_i = \dfrac{X_i - \mu}{\sigma}$

So $E[Z_i] = 0$ and $Var[Z_i] = 1$

Let $Y = \dfrac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$

②

$$M_Y(t) = M_{\frac{1}{\sqrt{n}}\sum Z_i}(t) = M_{\sum Z_i}(t/\sqrt{n})$$

$$= M_{Z_1}(t/\sqrt{n})\, M_{Z_2}(t/\sqrt{n}) \cdots M_{Z_n}(t/\sqrt{n})$$

$$= [M_{Z_1}(t/\sqrt{n})]^n$$

$$= \left(E[e^{t/\sqrt{n}\, Z_1}]\right)^n$$

$$= \left(E\left[1 + \frac{t}{\sqrt{n}} Z_1 + \frac{t^2 Z_1^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right]\right)^n$$

$$= \left[1 + 0 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right]^n$$

$$M_Y(t) = \left[1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right]^n$$

Find the limit as $n \to \infty$ ③

$$\ln M_Y(t) = n \ln\left[1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right]$$

$$= \frac{\ln\left[1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right]}{\frac{1}{n}} \quad \left(\frac{0}{0}\right)$$

$$\lim_{n\to\infty} \ln M_Y(t) =$$

$$\lim_{n\to\infty} \frac{\frac{1}{1+\frac{t^2}{2n}+O\left(\frac{1}{n^{3/2}}\right)} \cdot \left[\frac{-t^2}{2n^2} + O\left(\frac{1}{n^{5/2}}\right)\right]}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{t^2}{2n}+O\left(\frac{1}{n^{3/2}}\right)}\left[\frac{t^2}{2} + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

④

$$= \frac{t^2}{2}$$

So $\lim_{n\to\infty} M_Y(t) = e^{\frac{t^2}{2}}$

This is the m.g.f. of the standard normal distribution.

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$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu\right)$$

⑤

$$= \frac{1}{\sigma\sqrt{n}}\left(n\bar{X} - n\mu\right)$$

$$= \frac{n}{\sigma\sqrt{n}}(\bar{X}-\mu)$$

$$Y = \frac{\bar{X}-\mu}{(\sigma/\sqrt{n})}$$

Regardless of the original distribution of $X_1, .., X_n$,

$Y$ has a distribution that approaches the standard normal as $n \to \infty$

Central Limit Theorem

⑥

Regardless of the original distribution of $X_1, .., X_n$,

$\bar{X}$ will have an approximate normal distribution with mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$, provided that $n$ is large.

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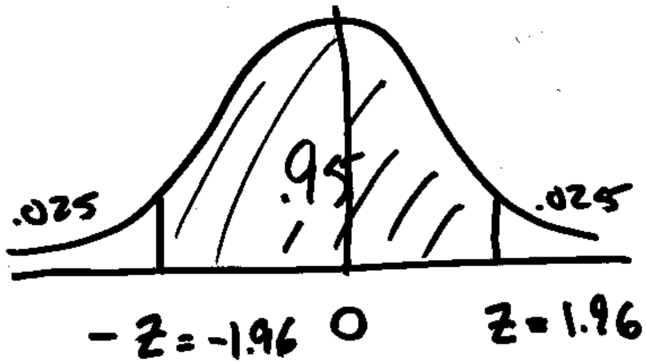
Goal: Use this theorem to estimate a population mean and give a margin of error.

⑦




$$P(-1.96 < Z < 1.96) = .95$$

$$P\left(-1.96 < \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < 1.96\right) \approx .95$$

$$-1.96 < \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < 1.96$$ true with 95% Confidence

⑧

$$-1.96\frac{\sigma}{\sqrt{n}} < \bar{x}-\mu < 1.96\frac{\sigma}{\sqrt{n}}$$

$$-\bar{x}-1.96\frac{\sigma}{\sqrt{n}} < -\mu < -\bar{x}+1.96\frac{\sigma}{\sqrt{n}}$$

$$\bar{x}+1.96\frac{\sigma}{\sqrt{n}} > \mu > \bar{x}-1.96\frac{\sigma}{\sqrt{n}}$$

With 95% confidence, we can state that the population mean, $\mu$, will fall with $\bar{x} \pm \underbrace{1.96\frac{\sigma}{\sqrt{n}}}_{\text{margin of error}}$

point estimate ($\bar{x}$)

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Problem: $\sigma$ is usually unknown. What happens if $s$ is used in its place?

$$\frac{\bar{X}-\mu}{s/\sqrt{n}} = \frac{\left[\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right] \leftarrow N(0,1)}{\dfrac{s}{\sigma} \leftarrow ?}$$

Facts: ① The random variables $\bar{x}$ and $s$ are independent

② $\dfrac{(n-1)s^2}{\sigma^2}$ has a $\chi^2$ distribution with $n-1$ d.f.

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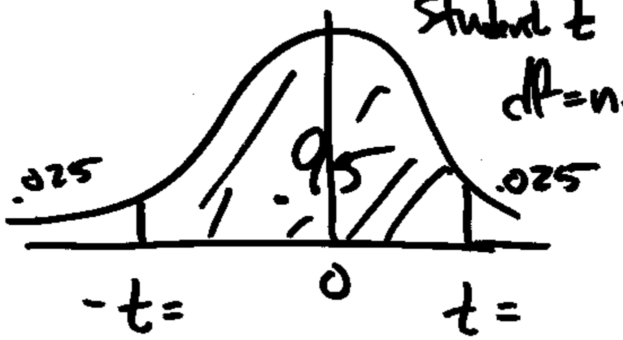
③ $\dfrac{Z}{\sqrt{\dfrac{\chi^2}{df}}}$ has a $t$ distribution

$$\frac{\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)s^2}{\sigma^2}\dfrac{1}{n-1}}} = \frac{\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}}{\dfrac{s}{\sigma}}$$

$$= \frac{\bar{x}-\mu}{s/\sqrt{n}}$$




Student t
$df = n-1$

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$\vdots$

$$\bar{x} \pm t\frac{s}{\sqrt{n}}$$

HW #9 p. 226 # 18
p. 252 # 24, 26

Stat 451 HW #8 p. 193 # 24, p. 205 # 42, 46, 52

24. $\text{Bino}(n=400,\ p=.5)$ $\quad \mu = np = 200 \quad \sigma = \sqrt{npq} = \sqrt{100} = 10$

a) $P(185 \le X \le 210) = P(184.5 \le X \le 210.5)$
$= P(-1.55 \le Z \le 1.05) = .8531 - .0606 = \underline{.7925}$

b) $P(X = 205) = P(204.5 \le X \le 205.5)$
$= P(.45 \le Z \le .55) = .7088 - .6736 = \underline{.0352}$

c) $P(X < 176) + P(X > 227)$
$= P(X \le 175) + P(X \ge 228)$
$= P(X \le 175.5) + P(X \ge 227.5)$
$= P(Z \le -2.45) + P(X \ge 2.75)$
$= .0071 + [1 - .9970] = \underline{.0101}$

42. $T \sim \text{Gamma}(\alpha = 2, \beta = \frac{1}{2})$ $\quad \lambda = \frac{1}{\beta} = 2$

a) $P(T > 1) = P(X \le 1)$, where $X \sim \text{Poisson}(\mu = \lambda t = 2)$
$= p(0) + p(1) = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!}\right) = 3e^{-2} = .4060$

So $P(T \le 1) = 1 - .4060 = \underline{.5940}$

b) $P(T \ge 2) = P(X \le 1)$, where $X \sim \text{Poisson}(\mu = 4)$
$= p(0) + p(1) = e^{-4}\left(\frac{4^0}{0!} + \frac{4^1}{1!}\right) = 5e^{-4} = \underline{.0916}$

46. For a single switch, $T \sim Exp(\beta = 2)$

$$P(T < 1) = F(1) = 1 - e^{-1/2}$$
$$= .3935$$

Then use $Bin(n = 100,\ p = .3935)$

+ find $P(X \le 30)$ Answer from Excel: $\underline{.033471}$

[= BINOMDIST (30, 100, 1 - EXP(-.5), 1)]

52. $\mu = \alpha\beta = 10 \quad \sigma = \sqrt{\alpha\beta^2} = \sqrt{\alpha}\,\beta = \sqrt{50}$

$$\Rightarrow \alpha = 2,\ \beta = 5$$

a) $P(T > 50) = P(X \le 1) \qquad X \sim \text{Poisson}\left(\lambda t = \frac{1}{5} \cdot 50 = 10\right)$

$$= e^{-10}\left(\frac{10^0}{0!} + \frac{10^1}{1!}\right) = 11e^{-10} = .000499$$

So $P(T \le 50) = 1 - P(T > 50) = \underline{.9995}$

b) $P(T > 10) = P(X \le 1) \qquad X \sim \text{Poisson}\left(\lambda t = \frac{1}{5} \cdot 10 = 2\right)$

$$= e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!}\right) = 3e^{-2} = .4060$$

So $P(T > 10) = 1 - .4060 = \underline{.5940}$