Chap 7 Sec 3 451

Moment generating functions

(1)

5-29

Defn: $M_X(t) = E[e^{tX}]$

$M_X(0) = E[e^0] = E[1] = 1$

$\frac{d}{dt} M_X(t) = \frac{d}{dt} E[e^{tX}]$

$= E[\frac{d}{dt} e^{tX}]$

$= E[e^{tX} X]$

$\frac{d}{dt} M_X(t) \Big|_{t=0} = E[X]$

(2)

$\frac{d^2}{dt^2} M_X(t) = E[\frac{d^2}{dt^2} e^{tX}]$

$= E[e^{tX} X^2]$

$\frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = E[X^2]$

Find the moment generating function for the exponential distribution.

$f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad x > 0$

$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\beta} e^{-x/\beta}\, dx$

$$= \frac{1}{\beta}\int_0^\infty e^{-x\left(\frac{1}{\beta}-t\right)}\,dx$$

③

$$= \frac{1}{\beta}\;\frac{e^{-x\left(\frac{1}{\beta}-t\right)}}{-\left(\frac{1}{\beta}-t\right)}\Bigg|_0^\infty$$

$$= 0 - \frac{1}{\beta}\,\frac{1}{-\left(\frac{1}{\beta}-t\right)} = \frac{1}{1-\beta t}$$

[provided $\frac{1}{\beta}-t>0$ OR $t<\frac{1}{\beta}$]

Check: $M_X(0) = 1$ ✓

$$M_X'(t) = -(1-\beta t)^{-2}(-\beta) = \beta(1-\beta t)^{-2}$$

$$M_X'(0) = \beta$$

④

$$M_X''(t) = \beta(-2)(1-\beta t)^{-3}(-\beta)$$

$$= 2\beta^2(1-\beta t)^{-3}$$

$$M_X''(0) = 2\beta^2 = E[X^2]$$

$$\sigma^2 = 2\beta^2 - \beta^2 = \beta^2$$

Find $M_X(t)$ for the binomial distribution

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x q^{n-x}$$

$$= \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x q^{n-x}$$

⑤

$$= (pe^t+q)^n \sum_{x=0}^{n} \binom{n}{x} \Big[\underbrace{\frac{pe^t}{pe^t+q}}_{p^*}\Big]^x \Big[\underbrace{\frac{q}{pe^t+q}}_{q^*}\Big]^{n-x}$$

$$= (pe^t+q)^n$$

Check: $M_X(0) = 1$ ✓

$$M_X'(t) = n(pe^t+q)^{n-1} pe^t$$

$$= npe^t(pe^t+q)^{n-1}$$

$$M_X'(0) = np$$

$$M_X''(t) = np\Big[e^t(n-1)(pe^t+q)^{n-2}pe^t + (pe^t+q)^{n-1}e^t\Big]$$

⑥

$$M_X''(0) = np\big[(n-1)p + 1\big]$$

$$= n(n-1)p^2 + np = E[X^2]$$

$$\sigma^2 = n(n-1)p^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np(-p+1)$$

$$= npq$$

Find the m.g.f. for the normal distribution

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

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$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 - 2\sigma^2 tx)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2(\mu + \sigma^2 t)x + \mu^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[x^2 - 2(\mu + \sigma^2 t)x + (\mu + \sigma^2 t)^2 - (\mu + \sigma^2 t)^2 + \mu^2]} dx$$

$$= e^{-\frac{1}{2\sigma^2}[-(\mu + \sigma^2 t)^2 + \mu^2]} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[x - (\mu + \sigma^2 t)]^2} dx}_{1}$$

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$$= e^{-\frac{1}{2\sigma^2}[-\cancel{\mu^2} - 2\mu\sigma^2 t - \sigma^4 t^2 + \cancel{\mu^2}]}$$

$$= e^{-\frac{1}{2\sigma^2}[-\sigma^2(2\mu t + \sigma^2 t^2)]}$$

$$= e^{\frac{1}{2}(2\mu t + \sigma^2 t^2)}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Notice the m.g.f. for the standard normal distribution is

$$M_X(t) = e^{\frac{1}{2}t^2}$$

⑨

Properties of the m.g.f.

Start with a random variable X.

Let $Y = X + c$

$$M_Y(t) = E[e^{tY}]$$
$$= E[e^{t(X+c)}]$$
$$= E[e^{tX} \cdot e^{tc}]$$
$$= e^{tc} E[e^{tX}]$$
$$= e^{tc} M_X(t)$$

⑩

Let $Y = aX$

$$M_Y(t) = E[e^{tY}]$$
$$= E[e^{taX}]$$
$$= M_X(ta)$$

Suppose X_1 and X_2 are independent random variables.

Let $Y = X_1 + X_2$

$$M_Y(t) = E[e^{tY}]$$
$$= E[e^{t(X_1+X_2)}]$$

$$= E[e^{tX_1} e^{tX_2}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tx_1} e^{tx_2} f(x_1, x_2)\, dx_1\, dx_2$$

independence

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{tx_1} e^{tx_2} g(x_1) h(x_2)\, dx_1}\, dx_2$$

$$= \int_{-\infty}^{\infty} \Big[e^{tx_2} h(x_2) \underbrace{\int_{-\infty}^{\infty} e^{tx_1} g(x_1)\, dx_1}_{M_{X_1}(t)} \Big] dx_2$$

$$= M_{X_1}(t)\, M_{X_2}(t)$$

Find the m.g.f. for the gamma distribution.

Think of the gamma random variable as the sum of α independent exponential random variables.

So $M_X(t) = \left(\frac{1}{1 - \beta t} \right)^{\alpha}$

Find the m.g.f. of the Poisson distribution.

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x e^{-\mu}}{x!}$$

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$$= \sum_{x=0}^{\infty} \frac{(\mu e^t)^x e^{-\mu}}{x!}$$

$$= e^{-\mu} e^{\mu e^t} \underbrace{\sum_{x=0}^{\infty} \frac{(\mu e^t)^x e^{-\mu e^t}}{x!}}_{1}$$

$$= e^{\mu(e^t - 1)}$$

Try the geometric, neg bino, uniform