Gamma function

451

(1)

5-24

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1), \quad \Gamma(1) = 1$$

For α an integer, $\Gamma(\alpha) = (\alpha-1)!$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-1/2} e^{-x}\, dx$$

let $y = \sqrt{2x}$ $\quad x = \frac{y^2}{2}$

$$\frac{dy}{dx} = \frac{1}{2}(2x)^{-1/2} \cdot 2$$

$$dy = (2x)^{-1/2} dx$$

$$dy = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{x}} dx$$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \sqrt{2}\, e^{-y^2/2}\, dy$$

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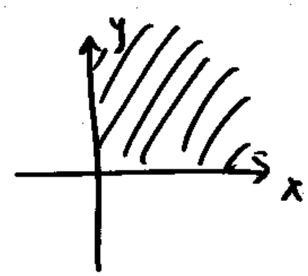
$$[\Gamma(\tfrac{1}{2})]^2 = \left[\int_0^\infty \sqrt{2}\, e^{-\frac{y^2}{2}} dy\right]\left[\int_0^\infty \sqrt{2}\, e^{-\frac{x^2}{2}} dx\right]$$

$$= 2\int_0^\infty\int_0^\infty e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy$$

$x = r\cos\theta, \quad y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta$

$$= 2\int_0^{\pi/2}\int_0^\infty e^{-\frac{1}{2}r^2} r\, dr\, d\theta$$



$$= 2\int_0^{\pi/2} -e^{-\frac{1}{2}r^2}\Big|_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} 1\, d\theta$$

$$= 2\,\theta\Big|_0^{\pi/2} = \pi$$

$\therefore \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

③

For example, $\Gamma(7/2)$

$$= \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\cdot\frac{3}{2}\Gamma(\frac{3}{2})$$

$$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$$

$$= \frac{15}{8}\sqrt{\pi}$$

Gamma distribution

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \qquad x>0$$

④

Suppose α is a positive integer.

Let X have a Poisson distribution with parameter λt.

The waiting time until the 1^{st} occurrence follows an exponential distribution with $\beta = \frac{1}{\lambda}$.

The waiting time until the α^{th} occurrence follows a Gamma distribution with parameters α and $\beta = \frac{1}{\lambda}$.

(5)

Suppose that the waiting time for α occurrences is longer than t.

Then the number of occurrences between times 0 and t is less than α, or less than or equal to $\alpha - 1$.

$$\boxed{P(\underset{\substack{\uparrow \\ \text{Gamma}(\alpha,\ \beta)}}{T} > \underline{t}) = P(\underset{\substack{\uparrow \\ \text{Poisson}(\underline{\lambda t})}}{X} \le \alpha - 1)}$$

$\beta = 1/\lambda$

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Example: Assume we have a Poisson process with an expected 5 occurrences per minute.

Find the probability that the waiting time for the 3rd occurrence is more than 30 seconds.

X = # occurrences in 30 seconds

$X \sim \text{Poisson}(2.5) \qquad \lambda t = 5 \cdot \frac{1}{2}$

T = waiting time until the 3rd occurrence

$T \sim \text{Gamma}(\alpha = 3, \beta = 1/5)$

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$$P(T > \tfrac{1}{2}) = P(X \le 2)$$

$$\| $$

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{(\frac{1}{5})^3 \Gamma(3)} x^2 e^{-5x} dx$$

$$p(0) + p(1) + p(2)$$

$$\|$$

$$e^{-2.5}\left[\frac{2.5^0}{0!} + \frac{2.5^1}{1!} + \frac{2.5^2}{2!} \right]$$

Same example:

Find the prob. that the waiting time for the 3^{rd} occurrences is between 30 secs and a minute.

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$$P(\tfrac{1}{2} < T < 1)$$

$$P(T > \tfrac{1}{2}) - P(T > 1)$$

$$P(T > \tfrac{1}{2}) = P(X \le 2)$$

$X \sim$ Poisson(2.5)

$$P(T > 1) = P(X \le 2)$$

$X \sim$ Poisson(5)

$$P(\tfrac{1}{2} < T < 1) = e^{-2.5}\left[\frac{2.5^0}{0!} + \frac{2.5^1}{1!} + \frac{2.5^2}{2!} \right] - e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} \right]$$

In Excel,

= POISSON(2, 2.5, 1) − POISSON(2, 5, 1)

If X has a Gamma distribution with parameters α and β, then X can be thought of as the sum of α independent random variables, each having an exponential distribution with parameter β.

So $\mu = \alpha\beta$

$\sigma^2 = \alpha\beta^2$

Consider a Gamma distribution with $\alpha = \frac{v}{2}$, where v is a positive integer, and $\beta = 2$

$$f(x) = \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} x^{\frac{v}{2}-1} e^{-\frac{1}{2}x} \qquad x>0$$

This is called the chi-squared distribution χ^2

v is called the "degrees of freedom."

Note: when $v=2$, this reduces to an exponential distribution

HW #8 p. 193 # 24

p. 205 # 42, 46, 52

due Thursday May 31.

(11)

Stat 451 HW #7 p.165 # 52, 58, 76 14 pts.

p. 186 # 4, 10, 12, 22 2 pts. each

52. NegBino$(k=2,\ p=\frac{1}{6})$ $p(x) = \binom{x-1}{k-1} p^k q^{x-k}$

$P(X=8) = \binom{7}{1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$

$= \underline{.0651}$

58. Poisson$(\mu=3)$ $p(x) = \dfrac{\mu^x e^{-\mu}}{x!}$

a) $p(5) = \dfrac{3^5 e^{-3}}{5!} = \underline{.1008}$

b) $P(X<3) = P(X \le 2) = p(0) + p(1) + p(2) = \underline{.4232}$

c) $P(X \ge 2) = 1 - P(X<2) = 1 - P(X \le 1) = 1 - [p(0) + p(1)]$

$= 1 - .1991 = \underline{.8009}$

76. a) Poisson$(\mu=2)$

$P(X \le 1) = p(0) + p(1) = \underline{.4060}$

b) Poisson$(\mu = \lambda t = 2 \cdot 5 = 10)$

$P(X \le 4) = p(0) + \dots + p(4) = \underline{.0293}$

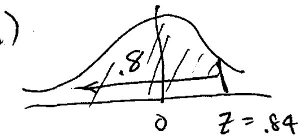
4. Normal$(\mu = 30, \sigma = 6)$

a) $P(X > 17) = P(Z > -2.17) = 1 - .0150 = \underline{.9850}$

b) $P(X < 22) = P(Z < -1.33) = \underline{.0918}$

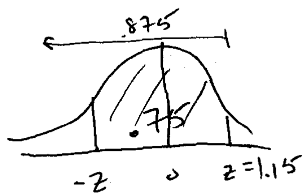
c) $P(32 < X < 41) = P(.33 < Z < 1.83)$
$= .9664 - .6293 = \underline{.3371}$

d)



$X = \mu + z\sigma$
$= 30 + .84(6) = \underline{35.04}$

e)



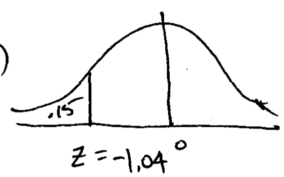
$X = 30 + 1.15(6) = \underline{36.9}$
$X = 30 - 1.15(6) = \underline{23.1}$

10. Normal$(\mu = 10, \sigma = .03)$

a) $P(X > 10.075) = P(Z > 2.5) = 1 - .9938 = \underline{.0062}$

b) $P(9.97 < X < 10.03) = P(-1 < Z < 1) = .8413 - .1587$
$= \underline{.6826}$

c)



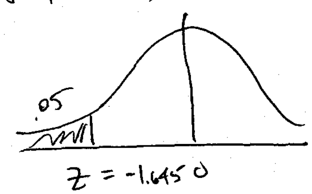
$X = 10 - 1.04(.03)$
$= \underline{9.9688}$

12. Normal$(\mu = 99.61, \sigma = .08)$

a) $P(99.5 < X < 99.7) = P(-1.375 < Z < 1.125)$

(I interpolated. Your answer may be slightly different) $= .8697 - .08455 = \underline{.78515}$

b)



$X = 99.61 - 1.645(.08)$
$= \underline{99.4784}$

22. Uniform$(A = 0, B = 10)$

a) $P(X > 7) = \frac{3}{10} = \underline{.3}$

b) $P(2 < X < 7) = \frac{5}{10} = \underline{.5}$