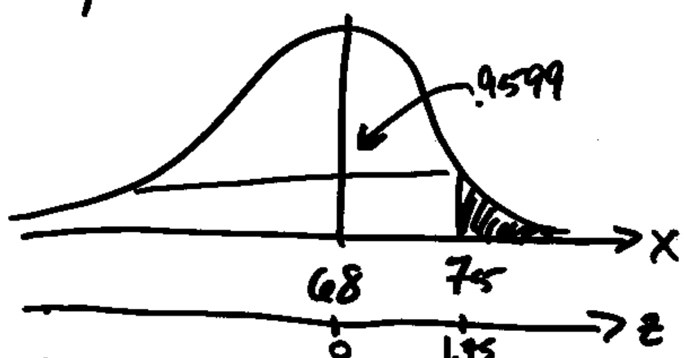
451

①

5-22

Normal distribution

Example: Suppose that speeds on I5 are normally distributed with $\mu = 68$ and $\sigma = 4$.



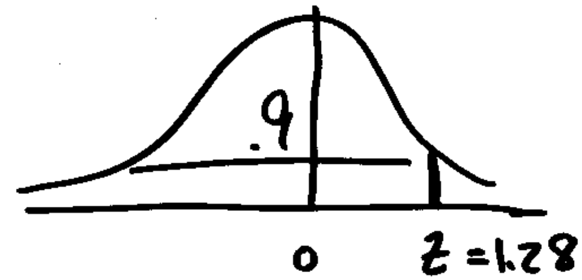
Find the percentage of speeds over 75.

$$P(X>75) = P(z > 1.75) = 1 - .9599 = 4.01\%$$

$$z = \frac{X-\mu}{\sigma} = \frac{75-68}{4} = 1.75$$

②

Find the 90th percentile of the speeds.



$$z = \frac{X-\mu}{\sigma} \iff X = \mu + \sigma z$$

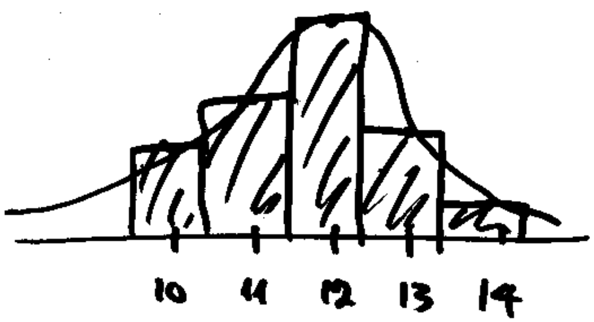
$$X = 68 + 1.28(4) = 73.12$$

(3)

Normal approximation to the binomial

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$



$$\underbrace{p(13)}_{\text{bino}} = \underbrace{P(12.5 < X < 13.5)}_{\text{normal}}$$

Continuity correction

(4)

$\mu = np, \quad \sigma = \sqrt{npq}$

Small example: Bino($n = 20$, $p = .6$)

Find $P(10 \le X \le 12)$

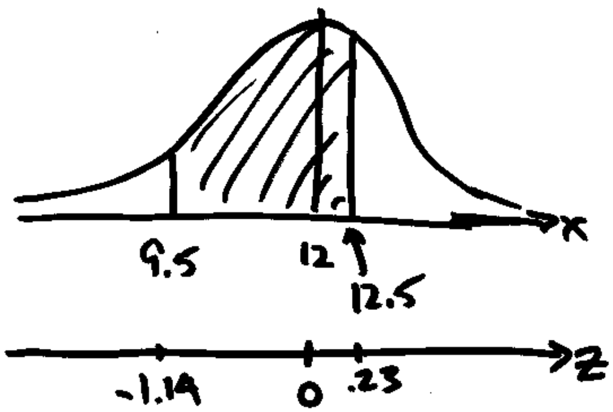
Normal approx:

$$P(9.5 \le X \le 12.5)$$

$$\mu = np = 20(.6) = 12$$

$$\sigma = \sqrt{npq} = \sqrt{20(.6)(.4)} = \sqrt{4.8} = 2.191$$

5



$$\frac{9.5-12}{2.191} = -1.14$$

$$\frac{12.5-12}{2.191} = .23$$

$.5910 - .1271$
$= .4639$

Approx. answer.

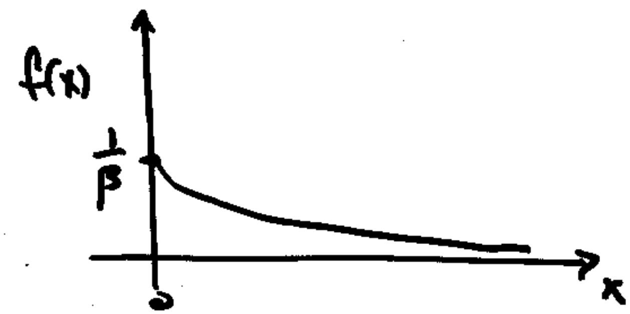
Exact answer: $p(10) + p(11) + p(12)$

$$= \binom{20}{10}(.6)^{10}(.4)^{10} + \binom{20}{11}(.6)^{11}(.4)^{9} + \binom{20}{12}(.6)^{12}(.4)^{8} = .4565$$

6

Exponential Distribution

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad x > 0$$



To probabilities, integrate:

$$P(a < X < b) = \int_a^b \frac{1}{\beta} e^{-x/\beta}\, dx$$

⑦

$$P(a \le X \le b) = F(b) - F(a)$$

What is $F(x)$?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{0}^{x} \frac{1}{\beta} e^{-t/\beta}\,dt$$

let $u = -t/\beta$

$du = -\frac{1}{\beta}\,dt$

⑧

$$F(x) = -\int_{0}^{-x/\beta} e^{u}\,du$$

$$= -e^{u}\Big|_{0}^{-x/\beta}$$

$$= -\left[e^{-x/\beta} - 1\right]$$

$$F(x) = 1 - e^{-x/\beta}$$

Example: Exponential with $\beta = 2$

Find the prob. that $1 \le X \le 5$

$$P(1 \le X \le 5) = F(5) - F(1)$$

$$= (1 - e^{-5/2}) - (1 - e^{-1/2})$$

$$= e^{-1/2} - e^{-5/2}$$ (9)

Find the mean:

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \frac{1}{\beta} e^{-x/\beta} dx$$

Let $u = x \quad dv = \frac{1}{\beta} e^{-x/\beta} dx$

$du = dx \quad v = -e^{-x/\beta}$

$$= uv\Big|_0^{\infty} - \int v\,du$$

$$= -xe^{-x/\beta}\Big|_0^{\infty} + \int_0^{\infty} e^{-x/\beta} dx$$

(10)

$$\lim_{x\to\infty} xe^{-x/\beta} \quad (\infty \cdot 0)$$

$$= \lim_{x\to\infty} \frac{x}{e^{x/\beta}} = \lim_{x\to\infty} \frac{1}{\frac{1}{\beta} e^{x/\beta}} = 0$$

So $\mu = \int_0^{\infty} e^{-x/\beta} dx$

$$= -\beta e^{-x/\beta} \Big|_0^{\infty}$$

$$= -\beta[0 - 1] = \beta$$

Similarly, $\sigma^2 = \beta^2$

Summary of exponential distribution:

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0$$

$$F(x) = 1 - e^{-x/\beta} = P(X \le x)$$

$$\mu = \beta, \quad \sigma^2 = \beta^2$$

Poisson connection:

$$p(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$p(0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

$= P(T > t)$ if T has an exponential distr. with $\beta = \frac{1}{\lambda}$

The gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

Let $u = x^{\alpha-1}$ $\quad dv = e^{-x} dx$

$du = (\alpha-1) x^{\alpha-2} dx \quad v = -e^{-x}$

$$\Gamma(\alpha) = \underbrace{-x^{\alpha-1} e^{-x} \Big|_0^\infty}_{0} + \int_0^\infty (\alpha-1) x^{\alpha-2} e^{-x} \, dx$$

$$= (\alpha-1) \int_0^\infty x^{\alpha-2} e^{-x} \, dx$$

$$\boxed{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)}$$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} \, dx = -e^{-x} \Big|_0^\infty$$

$$= -[0 - 1] = 1$$

(13)

$\Gamma(2) = 1 \cdot \Gamma(1) = 1 = 1!$

$\Gamma(3) = 2\,\Gamma(2) = 2 = 2!$

$\Gamma(4) = 3\,\Gamma(3) = 6 = 3!$

$\Gamma(5) = 4\,\Gamma(4) = 24 = 4!$

If α is a positive integer,
then $\Gamma(\alpha) = (\alpha - 1)!$

The gamma distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(14)

Notice that $\alpha = 1$ gives
the exponential distribution