451

1

5-15

Multivariate versions of the binomial & hypergeometric

Multinomial Distribution

Each trial has k possible outcomes

Run n independent trials

X_1 = # of times that the outcome was category 1

$\vdots$

X_k = # ...... category k

Note $X_1 + \dots + X_k = n$

$$p(x_1, x_2, \dots, x_k) = \binom{n}{x_1, x_2, \dots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

2

Multivariate Hypergeometric Distribution

Population of N items

a_1 are of type 1, a_2 are of type 2, ... a_k are of type k

$(a_1 + \dots + a_k = N)$

Take a sample of size n, WOR

X_1 = # of type 1 in sample

$\vdots$

X_k = # of type k in sample

$(X_1 + \dots + X_k = n)$

$$p(x_1, \dots, x_k) = \frac{\binom{a_1}{x_1}\binom{a_2}{x_2} \cdots \binom{a_k}{x_k}}{\binom{N}{n}}$$

Geometric Distribution

[Geometric Series: $a + ar + ar^2 + \ldots$
$= \frac{a}{1-r}$]

Run a sequence of Bernoulli trials

Assume that the trials are independent with constant success probability, p.

X = trial on which the 1st success occurs.

$$\boxed{\begin{array}{l} X = 1, 2, \ldots \\ p(x) = q^{x-1} p \end{array}}$$

$p(1) = p$
$p(2) = qp$
$p(3) = q^2 p$

Check to see if the probabilities sum to 1:

$$\sum_{x=1}^{\infty} q^{x-1} p \qquad \text{Let } y = x-1$$

$$= \sum_{y=0}^{\infty} q^y p = p + pq + pq^2 + \ldots$$

$$= \frac{p}{1-q} = \frac{p}{p} = 1 \checkmark$$

Find μ.

$$\mu = E[X] = \sum x p(x)$$

$$= \sum_{x=1}^{\infty} x q^{x-1} p$$

(5)

$$= \sum_{x=1}^{\infty} (x-1+1) q^{x-1} p$$

$$= \underbrace{\sum_{x=1}^{\infty} (x-1) q^{x-1} p}_{\text{Let } y = x-1} + \underbrace{\sum_{x=1}^{\infty} q^{x-1} p}_{1}$$

$$= \underbrace{\sum_{y=0}^{\infty} y q^{y} p} + 1$$

$$= \sum_{y=1}^{\infty} y q^{y} p + 1$$

$$= q \underbrace{\sum_{y=1}^{\infty} y q^{y-1} p}_{\mu} + 1$$

$$\mu = q\mu + 1 \Rightarrow p\mu = 1 \Rightarrow \boxed{\mu = 1/p}$$

(6)

Similar technique yields

$$\boxed{\sigma^2 = \frac{q}{p^2}}$$

Negative Binomial Distribution

or Pascal Distribution

Run a sequence of independent Bernoulli trials

X = trial on which the k^{th} success occurs

$X = k, k+1, \ldots$

⑦

$p(k) = p(k \text{ successes in a row})$

$= p^k$

$p(k+1) = P(k+1^{st} \text{ trial is a success} \cap$ in the 1^{st} k trials there was 1 failure and $k-1$ successes$)$

$= p \cdot \binom{k}{k-1} p^{k-1} q^1$

$= \binom{k}{k-1} p^k q$

$p(k+2) = P(k+2^{nd} \text{ trial is a success} \cap$ in the first $k+1$ tries there were $k-1$ successes$)$

$= p \cdot \binom{k+1}{k-1} p^{k-1} q^2 = \binom{k+1}{k-1} p^k q^2$

⑧

$$\boxed{p(x) = \binom{x-1}{k-1} p^k q^{x-k} \qquad x = k, k+1, \ldots}$$

If X_1 is the trial on which the 1^{st} success occurs,

X_2 is the number of trials past that one, where the next success occurs, etc.

Then

$$X = X_1 + \ldots + X_k$$

neg bin.

geometric

(9)

$$E[X] = E[X_1] + \dots + E[X_k]$$
$$= \frac{1}{p} + \dots + \frac{1}{p} = \frac{k}{p}$$

$$V[X] = V[X_1] + \dots + V[X_k]$$
$$= \frac{q}{p^2} + \dots + \frac{q}{p^2} = \frac{kq}{p^2}$$

<u>Examples</u>: X = trial on which the 5^{th} "head" occurs in a sequence of coin tosses.

Find $E[X] = \frac{k}{p} = \frac{5}{1/2} = 10$

$NegBino(k=5, p=1/2)$

(10)

Find the prob. that the 5^{th} head occurs on the 7^{th} toss.

Find $P(X=7) = p(7)$

$$= \binom{7-1}{5-1} (.5)^5 (.5)^2$$
$$= \binom{6}{4} (.5)^7$$
$$= .117$$

Each trial has a $\frac{1}{5}$ success probability. We continue until we get 10 successes.

$NegBino(k=10, p=1/5)$

$$\mu = E[X] = \frac{k}{p} = 50$$

(11)

Find the probability that the 10^{th} success occurs exactly at the 50^{th} trial

$$P(X=50) = p(50) = \binom{50-1}{10-1}(.2)^{10}(.8)^{40}$$

$$= \binom{49}{9}(.2)^{10}(.8)^{40}$$

$$= .028$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Let $y = \left(1+\frac{x}{n}\right)^n$

$$\ln y = n\ln\left(1+\frac{x}{n}\right)$$

$$= \frac{\ln(1+x/n)}{\frac{1}{n}}$$

(12)

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty}\frac{\frac{1}{\left(1+\frac{x}{n}\right)}\cdot\frac{\cancel{-}x}{\cancel{n^2}}}{\cancel{-}\frac{1}{\cancel{n^2}}} = x$$

$$\lim_{n\to\infty}\ln y = x$$

$$\therefore \lim_{n\to\infty} y = e^x$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

Start with $p(x) = \binom{n}{x} p^x q^{n-x}$

Take the limit as $n \to \infty$
and $p \to 0$ and $np = \mu$

$$\lim_{\substack{n\to\infty \\ p \to 0 \\ np = \mu}} \binom{n}{x} p^x q^{n-x} = \lim_{n\to\infty} \binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\mu^x}{n^x} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

$$= \frac{\mu^x}{x!} \lim_{n\to\infty} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x} \frac{n}{n} \frac{(n-1)}{n} \cdots \frac{(n-x+1)}{n}$$

$$= \frac{\mu^x}{x!} \frac{e^{-\mu}}{1} 1 \cdots 1$$

$$p(x) = \frac{\mu^x e^{-\mu}}{x!}$$ Poisson Distribution