Properties of $E[X]$, $V[X]$

451
1
5-1

$$E[X] = \sum_x x\,p(x)$$

or $\int_{-\infty}^{\infty} x\,f(x)\,dx$

$$E[aX] = \int_{-\infty}^{\infty} a\,x\,f(x)\,dx = a\int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= a\,E[X]$$

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\,f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dy\,dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(x,y)\,dy\,dx$$

$$= E[X] + E[Y]$$

2

Example: Find $E[2X-3Y+7]$

$$= 2E[X] - 3E[Y] + 7$$

Example: Suppose that $X_1, X_2, \ldots, X_{10}$ are random, all with the same mean μ.

Find $E[\bar{X}] = E\left[\frac{1}{10}(X_1 + \ldots + X_{10})\right]$

$$= \frac{1}{10}\left[E[X_1] + \ldots + E[X_{10}]\right]$$

$$= \frac{1}{10}[10\mu] = \mu$$

3

$$V[X] = \sigma^2 = E[(X-\mu)^2]$$

$$= E[X^2] - \mu^2 \qquad \mu = E[X]$$

$$V[aX] = E[(aX)^2] - (a\mu)^2$$

used the property of $E(X)$

$$= E[a^2X^2] - a^2\mu^2$$

$$= a^2[E[X^2] - \mu^2] = a^2 V(X)$$

4

$$V[X+Y] = E[(X+Y)^2] - (\mu_X+\mu_Y)^2$$

$$= E[X^2 + Y^2 + 2XY] - [\mu_X^2 + \mu_Y^2 + 2\mu_X\mu_Y]$$

$$= E[X^2] - \mu_X^2 + E[Y^2] - \mu_Y^2$$
$$+ 2[E[XY] - \mu_X\mu_Y]$$

$$= V(X) + V(Y) + 2Cov(X,Y)$$

$$= \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}$$

$$V(a) = 0$$

$$V[X+a] = V[X] + \cancel{V(a)}^{0} + 2\cancel{Cov(X,a)}^{0}$$

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$$\mathrm{Cov}(X,a) = E[Xa] - E[X]E(a)$$
$$= a\,E[X] - a\,E[X] = 0$$

Example : Find $V[2X-3Y+7]$

$$= V[2X-3Y]$$
$$= V(2X) + V(-3Y) + 2\,\mathrm{Cov}(2X,-3Y)$$
$$= 4V(X) + 9V(Y) + 2\,\mathrm{Cov}(2X,-3Y)$$

Need one more property to do this

$$= 4\,V(X) + 9V(Y) - 12\,\mathrm{Cov}(X,Y)$$

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$$\mathrm{Cov}(aX,bY)$$
$$= E[abXY] - E(aX)E(bY)$$
$$= ab\,E[XY] - aE[X]\,b\,E[Y]$$
$$= ab\,\mathrm{Cov}(X,Y)$$

What if X and Y are independent?

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} xy\; g(x)h(y)\,dy}\,dx$$

$$= \int_{-\infty}^{\infty}\left[x g(x)\int_{-\infty}^{\infty} y\, h(y)\,dy\right]dx$$

$$= \int_{-\infty}^{\infty} x g(x) \mu_Y \, dx$$

$$= \mu_Y \int_{-\infty}^{\infty} x g(x) \, dx$$

$$= \mu_Y \mu_X$$

Independence $\Rightarrow$ $E[XY] = \mu_X \mu_Y$

$\Rightarrow$ $Cov(X,Y) = 0$

Assuming independence,

$$V[X+Y] = V[X] + V[Y]$$

Example: Assume $X_1, \ldots, X_{10}$ are independent random variables, all with the same variance σ^2.

Find $V(\bar{X}) = V\left[\frac{1}{10}(X_1 + \ldots + X_{10})\right]$

$$= \frac{1}{100} V[X_1 + \ldots + X_{10}]$$

$$= \frac{1}{100}[\sigma^2 + \ldots + \sigma^2]$$

$$= \frac{1}{100} 10\sigma^2 = \frac{\sigma^2}{10}$$

$$\sigma_{\bar{X}} = \sqrt{\frac{\sigma^2}{10}} = \frac{\sigma}{\sqrt{10}}$$

HW #5 due May 8
(midterm day)

p. 134 # 58, 62, 70, 72

(9)

STAT 451 HW#4

p.101 #40, 44, 52 p.113 #10, 12, 26 p.122 #46, 50

p.101

40. $f(x,y) = \frac{2}{3}(x+2y) \quad 0 \le x \le 1, \; 0 \le y \le 1$

a) $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy = \frac{2}{3}\left[xy + y^2\right]_0^1$

$g(x) = \frac{2}{3}\left[x+1\right], \quad 0 \le x \le 1$

b) $h(y) = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1$

$h(y) = \frac{2}{3}\left[\frac{1}{2} + 2y\right], \quad 0 \le y \le 1$

c) $P[X < .5] = \int_0^{.5} \frac{2}{3}[x+1]\,dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^{.5}$

$= \frac{2}{3}\left[\frac{1}{8} + \frac{1}{2}\right] = \frac{2}{3}\left[\frac{5}{8}\right] = \frac{5}{12}$

44. $f(x,y) = k(x^2+y^2)$ $\quad 30 \le x < 50, \quad 30 \le y < 50$

a) Find k.

$$\int_a^b \int_c^d (x^2+y^2)\,dy\,dx = \int_a^b \left(x^2 y + \frac{y^3}{3}\right)\Big|_c^d dx$$

$$= \int_a^b \left[\left(dx^2 + \frac{d^3}{3}\right) - \left(cx^2 + \frac{c^3}{3}\right)\right] dx$$

$$= \int_a^b \left[(d-c)\,x^2 + \frac{d^3-c^3}{3}\right] dx$$

$$= (d-c)\frac{x^3}{3} + \frac{d^3-c^3}{3}x \Big|_a^b$$

$$= \left[\frac{(d-c)b^3}{3} + \frac{d^3-c^3}{3}b\right] - \left[\frac{(d-c)a^3}{3} + \frac{d^3-c^3}{3}a\right]$$

$$= \frac{(d-c)(b^3-a^3)}{3} + \frac{(d^3-c^3)(b-a)}{3}.$$

So $1 = k\left[\frac{(50-30)(50^3-30^3)}{3} + \frac{(50^3-30^3)(50-30)}{3}\right]$

So $k = \frac{3}{3{,}920{,}000}$

44b. $P[30 \le x < 40 \cap 40 \le y < 50]$

$$= k\left[\frac{(50-40)(40^3-30^3)}{3} + \frac{(50^3-40^3)(40-30)}{3}\right]$$

$$= k\left[\frac{370{,}000}{3} + \frac{610{,}000}{3}\right] = k\left(\frac{980{,}000}{3}\right)$$

$$= \frac{980{,}000}{3{,}920{,}000} = \frac{1}{4}$$

c) $P[30 \le x < 40 \cap 30 \le y < 40]$

$$= k\left[\frac{(40-30)(40^3-30^3)}{3} + \frac{(40^3-30^3)(40-30)}{3}\right]$$

$$= k\left[\frac{740{,}000}{3}\right] = \frac{740{,}000}{3{,}920{,}000} = .189$$

52. 3 coins tossed $X = \#\text{heads}$, $Y = \#\text{heads} - \#\text{tails}$

$p(x,y)$		y: -3	-1	1	3
x	0	1/8	0	0	0
	1	0	3/8	0	0
	2	0	0	3/8	0
	3	0	0	0	1/8

p. 113

10.

$f(x,y)$		y: 1	2	3	
x	1	.1	.05	.02	.17
	2	.1	.35	.05	.50
	3	.03	.1	.2	.33
		.23	.50	.27	1

Find μ_x & μ_y.

$$\mu_x = 1(.17) + 2(.5) + 3(.33) = 2.16$$

$$\mu_y = 1(.23) + 2(.5) + 3(.27) = 2.04$$

12. $f(x) = 2(1-x)$ $0<x<1$ x is in units of \$5000

Find average profit.

$$\mu = E[x] = \int_0^1 x \cdot 2(1-x)\,dx = 2\int_0^1 x - x^2\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3},$$

or \$1666.67

Omit #26 ***

26. $f(x,y) = 4xy$ $0 < x, y < 1$

$$\text{Find } E\left[\sqrt{x^2+y^2}\right] = \int_0^1 \underbrace{\int_0^1 (x^2+y^2)^{1/2} 4xy\,dy}_{\substack{\text{let } u = x^2+y^2 \\ du = 2y\,dy}}\,dx$$

$$= \int_0^1 \int_{x^2}^{x^2+1} 2x u^{1/2}\,du\,dx = \int_0^1 \frac{2x u^{3/2}}{3/2}\Bigg|_{x^2}^{x^2+1} dx$$

$$= \int_0^1 \frac{4}{3}x\left[(x^2+1)^{3/2} - x^3\right] dx = \int_0^1 \frac{4}{3}x(x^2+1)^{3/2} - \frac{4}{3}x^4\,dx$$

$$= \left(\frac{2}{3}\frac{(x^2+1)^{5/2}}{5/2} - \frac{4}{3}\frac{x^5}{5}\right)\Bigg|_0^1$$

$$= \left(\frac{4}{15}2^{5/2} - \frac{4}{15}\right) - \left(\frac{4}{15} - 0\right) = \frac{4}{15}\left[2^{5/2} - 2\right] = .975$$

p.122 46. Refers to p.102 #44 Find σ_{XY}

$$E[XY] = k\int_{30}^{50}\int_{30}^{50} xy(x^2+y^2)\,dy\,dx$$

$$= k\int_{30}^{50}\int_{30}^{50} x^3y + xy^3\,dy\,dx = k\int_{30}^{50} \frac{x^3y^2}{2} + \frac{xy^4}{4}\Big|_{30}^{50}\,dx$$

$$= k\int_{30}^{50}\left(\frac{x^3 50^2}{2} + \frac{x\,50^4}{4}\right) - \left(\frac{x^3 30^2}{2} + \frac{x\,30^4}{4}\right)\,dx$$

$$= k\int_{30}^{50} \tfrac{1}{2}(50^2-30^2)x^3 + \tfrac{1}{4}(50^4-30^4)x\,dx$$

$$= k\left[\frac{(50^2-30^2)x^4}{8} + \frac{(50^4-30^4)x^2}{8}\Big|_{30}^{50}\right]$$

$$= \frac{k}{8}\left[(50^2-30^2)(50^4-30^4) + (50^4-30^4)(50^2-30^2)\right]$$

$$= k[3176{,}000{,}000] = 1665.3061$$

$$E[X] = k\int_{30}^{50}\int_{30}^{50} x(x^2+y^2)\,dy\,dx$$

$$= k\int_{30}^{50} x^3y + \frac{xy^3}{3}\Big|_{30}^{50}\,dx$$

$$= k\int_{30}^{50} (50-30)x^3 + \tfrac{1}{3}(50^3-30^3)x\,dx$$

$$= k\left[(50-30)\frac{x^4}{4} + (50^3-30^3)\frac{x^2}{6}\right]_{30}^{50}$$

$$= k\left[\frac{50-30}{4}(50^4-30^4) + \frac{(50^3-30^3)}{6}(50^2-30^2)\right]$$

$$= k\left[27{,}200{,}000 + \frac{78{,}400{,}000}{3}\right]$$

$$= 40.8163$$

Similarly, $E[Y] = 40.8163$

So $\mathrm{Cov}(X,Y) = 1665.3061 - (40.8163)^2$

$= -.66$

50. $f(x) = 2(1-x) \qquad 0<x<1$

Find σ^2 and σ. We already know $\mu = \frac{1}{3}$ from p. 113 #12.

$$E[x^2] = \int_0^1 x^2 \cdot 2(1-x)\,dx = 2\int_0^1 x^2 - x^3\,dx$$

$$= 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{6}$$

$$\sigma^2 = E[x^2] - \mu^2 = \frac{1}{6} - \frac{1}{3}^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \text{ OR } .056$$

$$\sigma = \sqrt{\frac{1}{18}} \text{ OR } \frac{1}{3\sqrt{2}} \text{ OR } \frac{\sqrt{2}}{6} \text{ OR } .236$$