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①

4-26

Review:

$f(x,y)$ is the joint density

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

X & Y are independent if

$$f(x,y) = g(x)h(y) \quad \forall x,y$$

Defn: $f(x|y) = \dfrac{f(x,y)}{h(y)}$

Review: $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$

②

Defn: Expectation for pairs of random variables

$$E[w(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\,f(x,y)\,dx\,dy$$

Special cases:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f(x,y)\,dx\,dy$$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dx\,dy$$

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$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x\, f(x,y)\, dy\right] dx$$

$$= \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] dx$$

$$= \int_{-\infty}^{\infty} x\, g(x)\, dx$$

Defn: Covariance of X and Y

$$\sigma_{XY} = \mathrm{Cov}(X,Y) = E\left[(X-\mu_x)(Y-\mu_y)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y)\, f(x,y)\, dx\, dy$$

④

Alternate formula for σ_{XY} :

$$E\left[(X-\mu_x)(Y-\mu_Y)\right]$$

$$= E\left[XY - \mu_Y X - \mu_x Y + \mu_x\mu_Y\right]$$

$$= E[XY] - \mu_Y E[X] - \mu_x E[Y] + \mu_x\mu_Y$$

$$= E[XY] - \mu_x\mu_Y - \mu_x\mu_Y + \mu_x\mu_Y$$

$$= E[XY] - \mu_x\mu_Y$$

$$\sigma_{XY} = E[XY] - E[X]E[Y]$$

Defn: Correlation between X and Y

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{V(X)\,V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Fact: $-1 \le \rho_{XY} \le 1$

Example: $f(x,y) = 8xy$

$0 \le y \le x \le 1$

Find the correlation between X and Y

$f(x,y) = 8xy \quad 0 \le y \le x \le 1$

$g(x) = 4x^3 \quad 0 \le x \le 1 \quad \int_0^x 8xy\,dy$

$h(y) = 4y(1-y^2) \quad 0 \le y \le 1 \quad \int_y^1 8xy\,dx$

$E[X] = 4/5 = \int_0^1 x \cdot 4x^3\,dx$

$E[X^2] = 2/3 = \int_0^1 x^2 \cdot 4x^3\,dx$

$E[Y] = 8/15 = \int_0^1 y \cdot 4y(1-y^2)\,dy$

$E[Y^2] = 1/3 = \int_0^1 y^2 \cdot 4y(1-y^2)\,dy$

$E[XY] = 4/9 = \int_0^1 \int_0^x xy\,8xy\,dy\,dx$

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$$V(X) = E(X^2) - [E(X)]^2$$
$$= 2/3 - (4/5)^2 = 2/75$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$
$$= 1/3 - (8/15)^2 = \frac{11}{225}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= 4/9 - \frac{4}{5}\cdot\frac{8}{15} = \frac{4}{225}$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{4/225}{\sqrt{\frac{2}{75}\cdot\frac{11}{225}}}$$
$$= .492$$
$$= \frac{2}{3}\sqrt{66}$$