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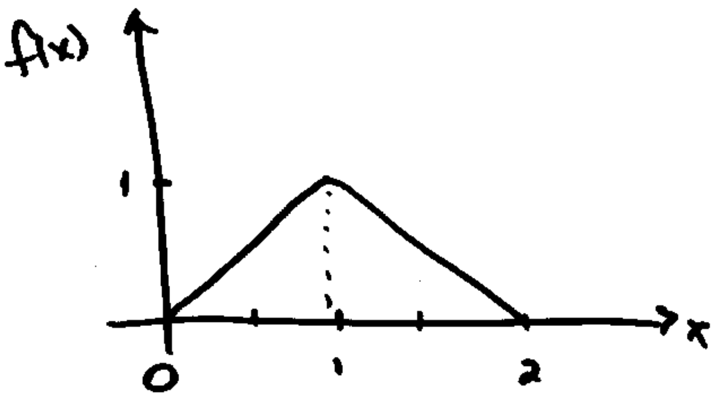
4-19

Continuous random variables

Example $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$



$$P(0 < X < 1) = \int_0^1 f(x)dx = \int_0^1 x\,dx$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = 1/2$$

2

$$P\left(\tfrac{1}{2} < X < \tfrac{3}{2}\right) = \int_{\frac{1}{2}}^{3/2} f(x)dx$$

$$= \int_{\frac{1}{2}}^{1} x\,dx + \int_{1}^{3/2} (2-x)dx$$

$$= 2\int_{\frac{1}{2}}^{1} x\,dx \quad \text{by symmetry}$$

$$= \cancel{2}\,\frac{x^2}{\cancel{2}}\Big|_{1/2}^{1} = 1 - \frac{1}{4} = 3/4$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

③

In our example,

$$\mu = \int_0^1 x \cdot x \, dx + \int_1^2 x(2-x)\,dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)\,dx$$

$$= \frac{x^3}{3}\Big|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2$$

$$= \frac{1}{3} - 0 + \left(4 - 8/3\right) - \left(1 - \frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$$

④

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

In particular,

$$E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

This is called the variance of X

& is written σ^2 or $Var(X)$

or $V(X)$

Alternate formula:

$$\int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\,dx$$

(5)

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu\int_{-\infty}^{\infty} x\, f(x)\,dx + \mu^2\int_{-\infty}^{\infty} f(x)\,dx$$

$$= E(X^2) - 2\mu\cdot\mu + \mu^2\cdot 1$$

$$\sigma^2 = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

Back to our example,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 x^2\, x\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \int_0^1 x^3\,dx + \int_1^2 (2x^2 - x^3)\,dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

(6)

$$= \frac{1}{4} - 0 + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{4}{3} - \frac{5}{12}$$

$$= \frac{3}{12} + \frac{16}{12} - \frac{5}{12}$$

$$= \frac{14}{12} = \frac{7}{6}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= \frac{7}{6} - 1^2 = \frac{1}{6}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{1}{6}}$$

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Cumulative distribution function (c.d.f.)

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$f(x) = \frac{d}{dx} F(x)$$

Example $f(x) = \begin{cases} kx & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$

1. Find k
2. Find $F(x)$

1. $1 = \int_0^1 kx\,dx$

$= \frac{kx^2}{2}\Big|_0^1 = \frac{k}{2}$

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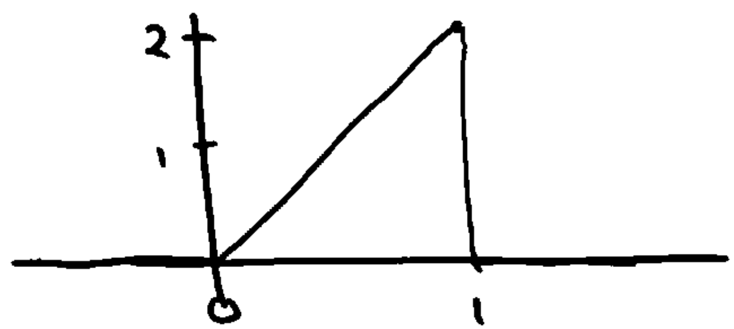
$\therefore k = 2$

$$f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$



$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \begin{cases} 0 & x \le 0 \\ \int_0^x 2t\,dt & 0<x<1 \\ 1 & x \ge 1 \end{cases}$$

(9)

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F(x) = P(X \le x)$$

Usage: $P(a < X \le b)$

$= F(b) - F(a)$

Return to discrete distributions

Consider 2 random variables, observed at the same time

(10)

Example: Roll 2 dice

Let X = minimum of the 2 dice

Y = maximum " " " "

$p(x,y)$	$Y=1$	2	3	4	5	6	
$X=1$	1/36	2/36	2/36	2/36	2/36	2/36	11/36
2	0	1/36	2/36	2/36	2/36	2/36	9/36
3	0	0	1/36	2/36	2/36	2/36	7/36
4	0	0	0	1/36	2/36	2/36	5/36
5	0	0	0	0	1/36	2/36	3/36
6	0	0	0	0	0	1/36	1/36
	1/36	3/36	5/36	7/36	9/36	11/36	1

This is a table of the joint distribution

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$$\sum_y p(x,y) = f(x)$$

$$\sum_x p(x,y) = g(y)$$

$$\sum_x \sum_y p(x,y) = 1$$

The conditional distribution of X, given Y is

$$f(x|y) = \frac{p(x,y)}{g(y)}$$

$$\left(\text{conditional} = \frac{\text{joint}}{\text{marginal}}\right)$$

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In the dice example,

Find $p(2,3) = 2/36$

This is the joint probability that $X=2$ and $Y=3$

Find $f(2) = 9/36$

This is the marginal probability that $X=2$.

Find $g(3) = 5/36$

This is the marginal probability that $Y=3$

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Find $f(2|3) = \dfrac{p(2,3)}{g(3)} = \dfrac{2/36}{5/36}$
$= 2/5$

This is the conditional probability that $X=2$, given that $Y=3$

<u>Example</u> 3 games

The team
- wins $\frac{1}{2}$ the time
- loses 40% "
- ties 10% "

$X = \#$ wins $\quad Y = \#$ losses

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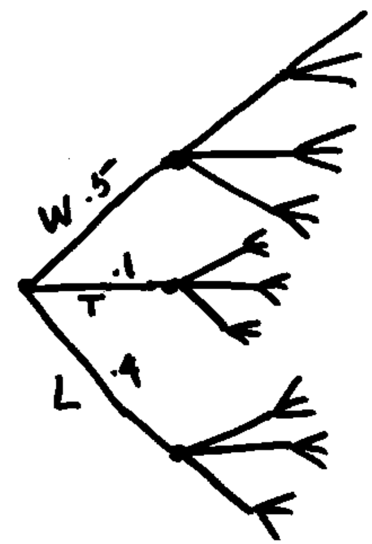
$p(x,y)$	Y = 0	1	2	3
X = 0	$(.1)^3$	$3(.4)(.1)^2$		
1				0
2			0	0
3		0	0	0



$$p(x,y) = \frac{3!}{x!\,y!\,(3-x-y)!} \cdot .5^x \, .4^y \, .1^{3-x-y}$$
for $0 \le x+y \le 3$