Stat 451 Homework #4

3.40 (a) $g(x) = \frac{2}{3}\int_0^1 (x+2y)\ dy = \frac{2}{3}(x+1)$, for $0 \le x \le 1$.

(b) $h(y) = \frac{2}{3}\int_0^1 (x+2y)\ dy = \frac{1}{3}(1+4y)$, for $0 \le y \le 1$.

(c) $P(X < 1/2) = \frac{2}{3}\int_0^{1/2} (x+1)\ dx = \frac{5}{12}$.

3.44 (a) $1 = k\int_{30}^{50}\int_{30}^{50}(x^2+y^2)\ dx\ dy = k(50-30)\left(\int_{30}^{50} x^2\ dx + \int_{30}^{50} y^2\ dy\right) = \frac{392k}{3}\cdot 10^4$.
So, $k = \frac{3}{392}\cdot 10^{-4}$.

(b) $P(30 \le X \le 40,\ 40 \le Y \le 50) = \frac{3}{392}\cdot 10^{-4}\int_{30}^{40}\int_{40}^{50}(x^2+y^2)\ dy\ dx$
$= \frac{3}{392}\cdot 10^{-3}(\int_{30}^{40} x^2\ dx + \int_{40}^{50} y^2\ dy) = \frac{3}{392}\cdot 10^{-3}\left(\frac{40^3-30^3}{3} + \frac{50^3-40^3}{3}\right) = \frac{49}{196}$.

(c) $P(30 \le X \le 40,\ 30 \le Y \le 40) = \frac{3}{392}\cdot 10^{-4}\int_{30}^{40}\int_{30}^{40}(x^2+y^2)\ dx\ dy$
$= 2\frac{3}{392}\cdot 10^{-4}(40-30)\int_{30}^{40} x^2\ dx = \frac{3}{196}\cdot 10^{-3}\frac{40^3-30^3}{3} = \frac{37}{196}$.

3.52 A tabular form of the experiment can be established as,

Sample Space	x	y
HHH	3	3
HHT	2	1
HTH	2	1
THH	2	1
HTT	1	-1
THT	1	-1
TTH	1	-1
TTT	0	-3

So, the joint probability distribution is,

	$f(x,y)$	x: 0	1	2	3
y	-3	1/8			
	-1		3/8		
	1			3/8	
	3				1/8

4.10 $\mu_X = \sum xg(x) = (1)(0.17) + (2)(0.5) + (3)(0.33) = 2.16$,
$\mu_Y = \sum yh(y) = (1)(0.23) + (2)(0.5) + (3)(0.27) = 2.04$.

4.12 $E(X) = \int_0^1 2x(1-x)\ dx = 1/3$. So, $(1/3)(\$5,000) = \$1,667.67$.

4.26 $E(Z) = E(\sqrt{X^2+Y^2}) = \int_0^1 \int_0^1 4xy\sqrt{x^2+y^2}\ dx\ dy = \frac{4}{3}\int_0^1 [y(1+y^2)^{3/2} - y^4]\ dy$
$= 8(2^{3/2}-1)/15 = 0.9752.$

4.46 From previous exercise, $k = \left(\frac{3}{392}\right) 10^{-4}$, and $g(x) = k\left(20x^2 + \frac{98000}{3}\right)$, with
$\mu_X = E(X) = \int_{30}^{50} xg(x)\ dx = k\int_{30}^{50} \left(20x^3 + \frac{98000}{3}x\right)\ dx = 40.8163.$
Similarly, $\mu_Y = 40.8163$. On the other hand,
$E(XY) = k\int_{30}^{50}\int_{30}^{50} xy(x^2+y^2)\ dy\ dx = 1665.3061.$
Hence, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = 1665.3061 - (40.8163)^2 = -0.6642.$

4.50 $E(X) = 2\int_0^1 x(1-x)\ dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$ and
$E(X^2) = 2\int_0^1 x^2(1-x)\ dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}$. Hence,
$Var(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$, and $\sigma = \sqrt{1/18} = 0.2357.$