Estimation method 2: <u>Maximum Likelihood Est.</u> Stat 451

(MLE) 3-1-18

(1)

Start with the joint probability mass function (discrete)

or joint prob. density fn. (continuous case)

Assuming $X_1, \ldots, X_n$ are independent, identically distributed,

$$f(x_1, \ldots, x_n) = f(x_1) f(x_2) \ldots f(x_n)$$
$$= \prod_{i=1}^{n} f(x_i)$$

Treat this as function of the unknown parameter.

In this context, it is called the "likelihood" function.

$$\mathcal{L}(\theta) = \text{lik}(\theta) = \prod_{i=1}^{n} f(x_i)$$ (2)

Find the value of θ that maximizes $\mathcal{L}(\theta)$

<u>Example</u>: $f(x) = \lambda e^{-\lambda x}$

$$\mathcal{L}(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \left(\lambda e^{-\lambda x_i}\right)$$
$$= \lambda^n e^{-\lambda \Sigma x_i}$$

"log likelihood"

$$\ell(\lambda) = \ln \mathcal{L}(\lambda) = n \ln \lambda - \lambda \Sigma x_i$$

③

$$\ell'(\lambda) = \frac{n}{\lambda} - \sum x_i \overset{set}{=} 0$$

$$\frac{n}{\lambda} = \sum x_i \qquad \hat{\lambda}_{MLE} = \frac{n}{\sum x_i} = \frac{1}{\bar{X}}$$

Example $p(x) = \frac{\alpha^x e^{-\alpha}}{x!}$

$$\mathcal{L}(\alpha) = \prod_{i=1}^{n} p(x_i) = \prod_{i=1}^{n} \frac{\alpha^{x_i} e^{-\alpha}}{x_i!}$$

$$= \frac{\alpha^{\sum x_i} e^{-n\alpha}}{\prod_{i=1}^{n} x_i!}$$

$$\ell(\alpha) = \sum x_i \ln \alpha - n\alpha - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

④

$$\ell'(\alpha) = \frac{\sum x_i}{\alpha} - n \overset{set}{=} 0$$

$$\hat{\alpha}_{MLE} = \frac{\sum x_i}{n} = \bar{x}$$

Example: $f(x) = \frac{1}{b} \quad 0 < x < b$

Find both the MOM & MLE estimators

MOM: Equate $\bar{x}$ with μ

Set $\bar{x} = \frac{b}{2} \quad \therefore \hat{b}_{MOM} = 2\bar{x}$

MLE: $\mathcal{L}(b) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{b} \quad 0<x_i<b$ (5)

$$= \frac{1}{b^n} \quad 0<x_i<b \ \forall i$$

Note that $\mathcal{L}(b) \uparrow$ as $b \to 0$

So, to maximize $\mathcal{L}(b)$, make b as small as possible.

But b cannot be smaller than the $\max\{x_i\}$

$$\therefore \hat{b}_{MLE} = \max\{x_i\}$$

Back to the subject of confidence intervals. (6)

for μ: $\bar{x} \pm z\frac{\sigma}{\sqrt{n}}$ OR $\bar{x} \pm t\frac{s}{\sqrt{n}}$

for p: $\hat{p} \pm z\sqrt{\frac{\hat{p}\hat{q}}{n}}$

What if you are given a desired margin of error?

Find the necessary sample size.

for μ: Set $E = z\frac{\sigma}{\sqrt{n}}$ & solve for n

$$n = \left(\frac{z\sigma}{E}\right)^2$$

(7)

Example: In a population of heights, the standard deviation is known to be 3". How large a sample is needed, to estimate the population to within $\frac{1}{2}$", with 95% confidence?

$$n = \left(\frac{Z\sigma}{E}\right)^2 = \left(\frac{1.96(3)}{.5}\right)^2 = 138.3$$

Answer: $\underline{n = 139}$

(8)

What if σ is unknown?

Conduct a pilot study + estimate σ by s

$$\cancel{n = \left(\frac{ts}{E}\right)^2}$$

Can't solve, since t depends on $df = n-1$

Use $n = \left(\frac{Zs}{E}\right)^2$

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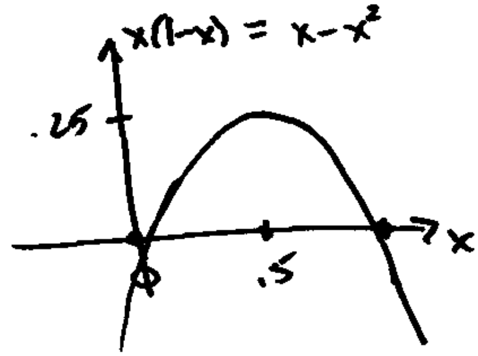
for p: Set $E = Z\sqrt{\frac{\hat{p}\hat{q}}{n}}$

$$n = \frac{Z^2\hat{p}\hat{q}}{E^2}$$

Since p and q are unknown, do a pilot study

(9)

$\hat{p}\hat{q}$ looks like $x(1-x)$



So $n = \dfrac{z^2 \hat{p}\hat{q}}{\varepsilon^2} \le \dfrac{z^2 (.5)(.5)}{\varepsilon^2}$

In a national election, to estimate the proportion favoring candidate A with 95% confidence and .03 margin of error, how large a sample is needed?

(10)

$$n = \frac{1.96^2 (.25)}{.03^2} = 1068$$

HW #7 due 3/16/18

p.269 # 2, 3, 10, 19

2. Analysis of the venom of 7 eight-day-old worker bees yielded the following observations on histamine content in nanograms: 649, 832, 418, 530, 384, 899, 755.

(a) Construct by hand a 90% CI for the true mean histamine content for all worker bees of this age. What assumptions, if any, are needed for the validity of the CI?

(b) The true mean histamine content will be in the CI you constructed in part (a) with probability 90%. True or false?

3. For a random sample of 50 measurements of the breaking strength of cotton threads, $\overline{X} = 210$ grams and $S = 18$ grams.

(a) Obtain an 80% CI for the true mean breaking strength. What assumptions, if any, are needed for the validity of the CI?

(b) Would the 90% CI be wider than the 80% CI constructed in part (a)?

(c) A classmate offers the following interpretation of the CI you obtained in part (a): We are confident that 80% of all breaking strength measurements of cotton threads will be within the calculated CI. Is this interpretation correct?

10. A health magazine conducted a survey on the drinking habits of young adult (ages 21–35) US citizens. On the question "Do you drink beer, wine, or hard liquor each week?" 985 of the 1516 adults interviewed responded "yes."

(a) Find a 95% confidence interval for the proportion, p, of young adult US citizens who drink beer, wine, or hard liquor on a weekly basis.

(b) The true proportion of young adults who drink on a weekly basis lies in the interval obtained in part (a) with probability 0.95. True or false?

19. Kingsford's regular charcoal with hickory is available in 15.7-lb bags. Long-standing uniformity standards require the standard deviation of weights not to exceed 0.1 lb. The quality control team uses daily samples of 35 bags to see if there is evidence that the standard deviation is within the required limit. A particular day's sample yields $S = 0.117$. Make a 95% CI for σ. Does the traditional value of 0.1 lie within the CI?