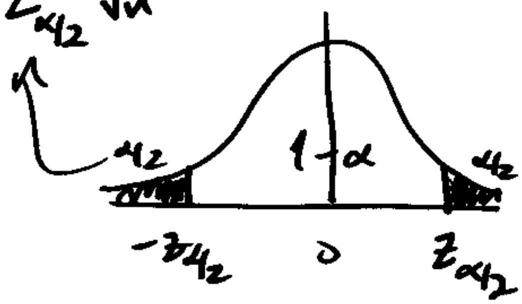
Stat 451
2-27-18

From last time:

① The confidence interval for μ was

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$



$\alpha = .95$
$\Rightarrow z = 1.96$

$\alpha = .99$
$\Rightarrow z = 2.576$

$\alpha = .90$
$\Rightarrow z = 1.645$

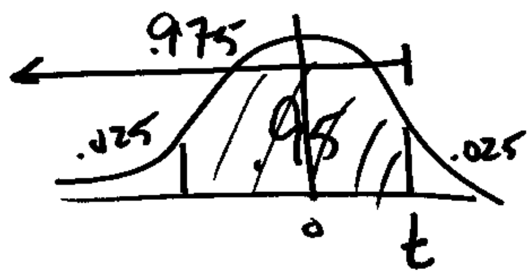
But usually, σ is unknown.
s can be used as an estimate of σ, provided that n is sufficiently large

② If n is not large (say $n < 30$)

Then use $\bar{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$

df = degrees of freedom $= n-1$

Student's t table



Problem from last time:

$\bar{x} = 70$ $n = 25$
$s = 3$ 95% C.I. for μ

$$70 \pm (2.064) \frac{3}{\sqrt{25}}$$

$$70 \pm 1.2384$$

③

New situation:

Run binomial trials

n known $\qquad X = \#$ successes

p unknown. $\qquad = X_1 + X_2 + \dots + X_n$,

where $X_i = \begin{cases} 0 & \text{failure} \\ 1 & \text{success} \end{cases}$

So $\frac{X}{n}$ is actually an average

By the Central Limit Theorem, $\frac{X}{n}$ will be approximately normal for sufficiently large n.

④

$$E\left[\frac{X}{n}\right] = \frac{1}{n} E[X] = \frac{1}{n} np = p$$

$$V\left[\frac{X}{n}\right] = \frac{1}{n^2} V[X] = \frac{1}{n^2} npq = \frac{pq}{n}$$

Define $\hat{p} = \frac{X}{n}$

Then $$\frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \sim N(0,1)$$

$\therefore$ A confidence interval for p is

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

(5)

<u>Example</u>: Take a sample of 40 PSU students. Find 21 are female. Find a 95% C.I. for the true percentage of female PSU students.

$n = 40 \quad X = 21$

$\hat{p} = \frac{X}{n} = .525$

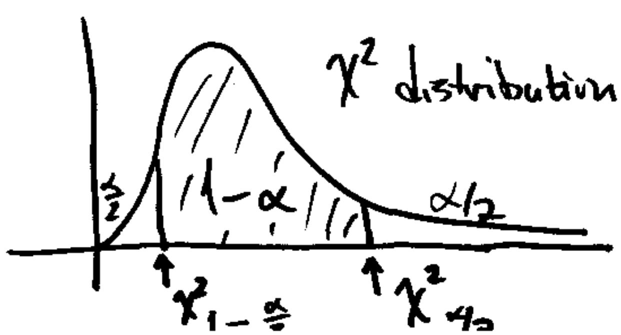
$.525 \pm 1.96\sqrt{\frac{(.525)(.475)}{40}}$

$.525 \pm .155$

(6)

Confidence interval for σ^2 or for σ

Fact: $\frac{(n-1)s^2}{\sigma^2}$ is a random variable which has a chi-squared (χ^2) distribution with $n-1$ df, provided that the sample came from a normally-distributed population.



(7)

$$P\left(\chi^2_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2} \right) = 1-\alpha$$

$$\chi^2_{1-\frac{\alpha}{2}} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2}$$ is an event whose probability is $1-\alpha$

$$\frac{1}{\chi^2_{1-\alpha/2}}\,(n-1)s^2 > \frac{\sigma^2}{\cancel{(n-1)s^2}} > \frac{1}{\chi^2_{\alpha/2}}\,(n-1)s^2$$

$$\boxed{\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}$$

For σ, take $\sqrt{\ }$ of both sides

(8)

<u>Example</u>: In a sample of 20 measurements of items produced on an assembly line, the sample standard deviation is 2mm. Assume that the measurements are normally distributed. Find a 95% C.I. for the population standard deviation.

$$\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}$$

$n = 20 \quad df = 19$ (9)

$s = 2$

$\alpha = 1 - \text{conf. level} = .05$

$\chi^2_{.025} = 32.852$

$\chi^2_{.975} = 8.907$

$$\sqrt{\frac{19(2^2)}{32.852}} < \sigma < \sqrt{\frac{19(2^2)}{8.907}}$$

$(1.52, 2.92)$ is our 95% C.I. for σ

(10)

What if you need to estimate a parameter other than μ, p, or σ?

Example: $f(x) = \lambda e^{-\lambda x}$

Take a sample of size n. Find a estimator of λ.

Method 1: Method of moments (MoM)

Set the 1st sample moment $(\bar{x})$ equal to the 1st population moment (μ) and solve for the parameter of interest.

Recall that $E[X] = \frac{1}{\lambda}$ for the exponential distribution. (11)

MOM: Set $\bar{X} = \frac{1}{\lambda}$ & solve for λ

$$\therefore \hat{\lambda}_{MOM} = \frac{1}{\bar{X}}$$

Example: $p(x) = \frac{\alpha^x e^{-\alpha}}{x!}$ Recall $E[X] = \alpha$

MOM: Set $\bar{X} = \alpha$ & solve for α

$$\therefore \hat{\alpha} = \bar{X}$$

Example: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ (12)

Estimate both μ and σ^2.

MOM now requires that you equate the 2^{nd} sample moment to 2^{nd} population moment.

	Sample	population
1^{st} moment	$\bar{X}$	μ
2^{nd} moment	$\frac{1}{n}\sum X_i^2$	$E[X^2]$

MOM says set $\bar{X} = \mu$

& $\frac{1}{n}\sum X_i^2 = E[X^2]$ & solve

for μ & σ^2

$\sigma^2 = V[X] = E[X^2] - \mu^2$

$E[X^2] = \sigma^2 + \mu^2$

Set $\begin{cases} \bar{X} = \mu \\ \frac{1}{n}\sum X_i^2 = \sigma^2 + \mu^2 \end{cases}$ $\therefore \hat{\mu} = \bar{X}$

$\frac{1}{n}\sum X_i^2 = \sigma^2 + \bar{X}^2$ $\therefore \hat{\sigma}^2 = \frac{1}{n}\sum X_i^2 - \bar{X}^2$

$= \frac{n-1}{n} S^2$