Stat 451

2-22-18

1

Let $X_1, X_2, \ldots, X_n$ be <u>independent</u> random variables, each having the same mean μ and the same variance σ^2.

Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$E[\overline{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n} n\mu = \mu$$

2

$$V[\overline{X}] = V\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\left[\sum_{i=1}^{n} V[X_i] + \cancel{\text{covar.}}\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$\therefore \mu_{\overline{X}} = \mu \quad \text{and} \quad \sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$$

<u>Example 1</u>

Place slips of paper in a hat, numbered 1, 2, 3, 4.

Draw 1.

<u>Example 2</u>

Draw 2 slips <u>with replacement</u> + average them.

Example 1

x	$p(x)$
1	.25
2	.25
3	.25
4	.25

$$\mu = \sum x\,p(x) = 1(.25) + 2(.25) + 3(.25) + 4(.25) = 2.5$$

$$E[X^2] = \sum x^2 p(x) = 1^2(.25) + 2^2(.25) + 3^2(.25) + 4^2(.25) = \frac{15}{2}$$

Example 2

X_2

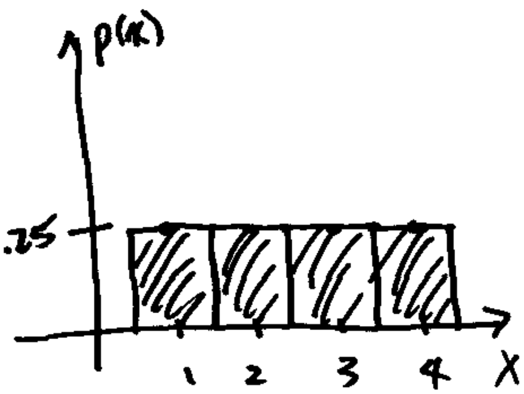
$\bar{X}$	1	2	3	4
X_1 1	1	1.5	2	2.5
2	1.5	2	2.5	3
3	2	2.5	3	3.5
4	2.5	3	3.5	4

$\bar{x}$	$p(\bar{x})$
1	1/16
1.5	2/16
2	3/16
2.5	4/16
3	3/16
3.5	2/16
4	1/16

$$\mu_{\bar{x}} = \sum \bar{x}\,p(\bar{x}) = 1\left(\tfrac{1}{16}\right) + 1.5\left(\tfrac{2}{16}\right) + \ldots = 2.5$$

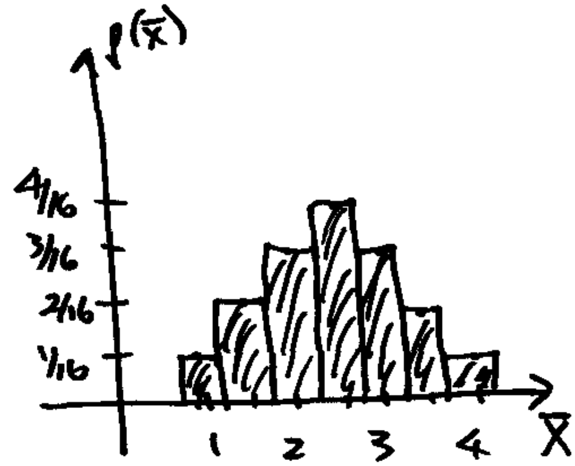
Example 1

$$\sigma^2 = E[X^2] - \mu^2 = \frac{15}{2} - 2.5^2 = 1.25$$



Example 2

$$\sigma^2 = .625$$



(5)

Central Limit Theorem

(iid)

If $X_1, \ldots, X_n$ are independent, identically distributed random variables, each having mean μ and variance σ^2, then as $n \to \infty$, the random variable $\dfrac{\bar{X} - \mu}{(\sigma/\sqrt{n})}$ has a distribution approaching $N(0,1)$.

Consequence: For large values of n,

$\bar{X}$ is approximately normal with $\mu_{\bar{X}} = \mu$ & $\sigma^2_{\bar{X}} = \dfrac{\sigma^2}{n}$

(6)

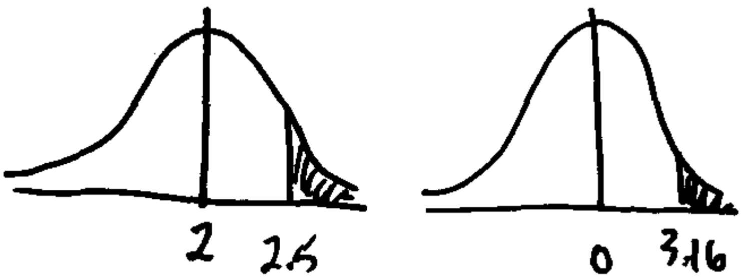
Example: A certain electronic component has a lifetime whose mean is 2 years (μ) & whose standard deviation is .5 years (σ). The distribution is skewed to the right.

Find the probability that a component lasts longer than 2.5 years. $P(X > 2.5)$ Not enough info

Observe 10 components. Assume they operate independently. Find the probability that their average lifetime is > 2.5 yrs.

7

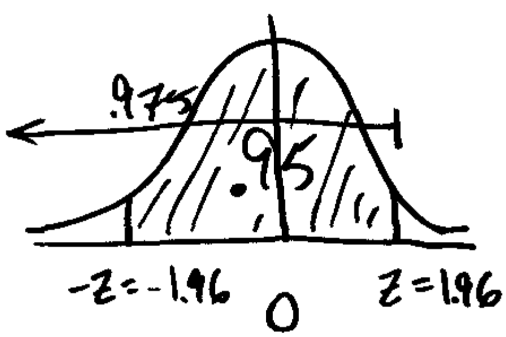
$$P(\bar{X} > 2.5) = P\left(Z > \frac{2.5-2}{(.5/\sqrt{10})}\right) = P(Z > 3.16)$$



$= .0008$

Example: In a sample of 25 people, we find an average height of 70" and a standard deviation of 3". Estimate the average height in the population & give a margin of error.

8



$$P(-1.96 < Z < 1.96) = .95$$

Central Limit Theorem said that $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \approx N(0,1)$

So $P\left(-1.96 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.96\right) \approx .95$

that is, $-1.96 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.96$ is an event whose probability is .95

$$-1.96\frac{\sigma}{\sqrt{n}} < \bar{x}-\mu < 1.96\frac{\sigma}{\sqrt{n}}$$

$$-\bar{x}-1.96\frac{\sigma}{\sqrt{n}} < -\mu < -\bar{x}+1.96\,\sigma/\sqrt{n}$$

$$\bar{x}+1.96\frac{\sigma}{\sqrt{n}} > \mu > \bar{x}-1.96\,\sigma/\sqrt{n}$$

$$\boxed{\bar{x}-1.96\,\sigma/\sqrt{n} < \mu < \bar{x}+1.96\,\sigma/\sqrt{n}}$$

True with 95% Confidence

This says that, if $\bar{x}$ is used to estimate μ, then, with 95% confidence, it's off by at most $1.96\,\sigma/\sqrt{n}$

95% Confidence interval for μ is

$$\underset{\substack{\uparrow\\ \text{estimate}\\ \text{of } \mu}}{\bar{x}} \pm \underbrace{1.96\,\sigma/\sqrt{n}}_{\substack{\uparrow\\ \text{margin}\\ \text{of error}}} = 70 \pm 1.96\frac{3}{\sqrt{25}}$$

Note: s was used since σ wasn't known

$$= 70 \pm 1.176$$

$\boxed{\text{HW \#6 p.228 \#5, \#10a}}$

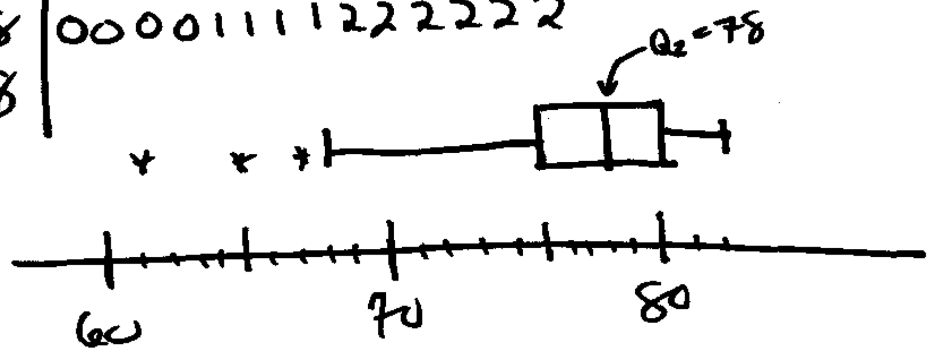
due 3/1

Exam results: 82 points max

6 | 1
6 | 578899
7 | 124
7 | 555666667788889999
8 | 00001111222222
8 |

$n = 43$

$\bar{x} = 76.42 / 82$
$= 93.2\%$



5. Two towers are constructed, each by stacking 30 segments of concrete vertically. The height (in inches) of a randomly selected segment is uniformly distributed in the interval (35.5, 36.5). A roadway can be laid across the 2 towers provided the heights of the 2 towers are within 4 inches of each other. Find the probability that the roadway can be laid. Be careful to justify the steps in your argument, and state whether the probability is exact or approximate.

10. A batch of 100 steel rods passes inspection if the average of their diameters falls between 0.495 cm and 0.505 cm. Let μ and σ denote the mean and standard deviation, respectively, of the diameter of a randomly selected rod. Answer the following questions assuming that $\mu = 0.503$ cm and $\sigma = 0.03$ cm.

(a) What is the (approximate) probability the inspector will accept (pass) the batch?