The Exponential distribution, continued

Stat 451

2-6-18

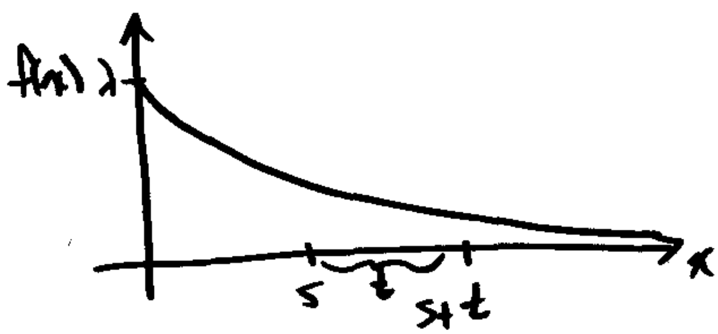
1

Recall: $f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

For $x > 0$
$P(X > x) = 1 - P(X \le x) = 1 - F(x) = e^{-\lambda x}$

Find $P(X > s+t \mid X > s)$



2

Recall: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

This is the memory-less property of the exponential distribution

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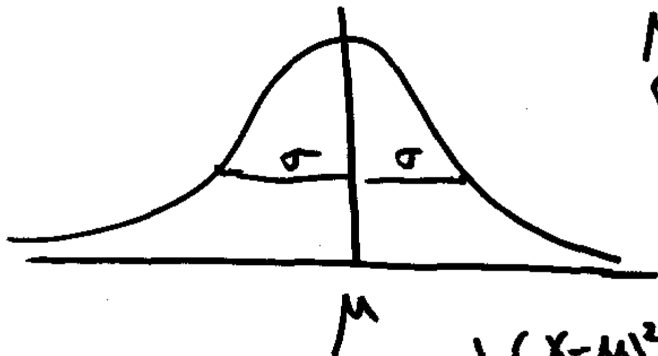
The Normal Distribution



μ is the mean
σ is the standard deviation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

If $\mu = 0$ and $\sigma = 1$, then $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

This is the standard normal distribution

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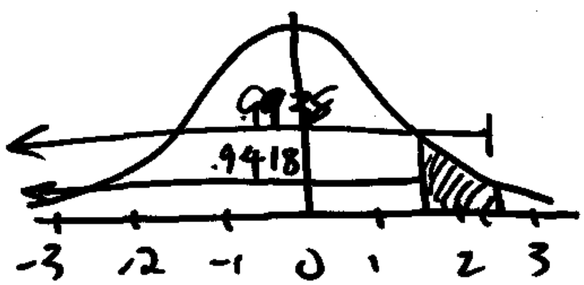
Ex: In the standard normal distribution, find the probability of seeing a value between 1.57 and 2.5.



$$P(1.57 < Z < 2.5) = .9938 - .9418 = .052$$

(5)

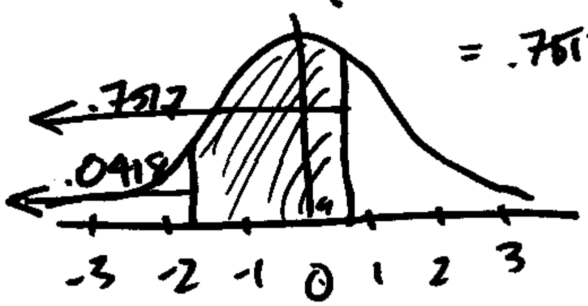
2 important facts about the standard normal

(1) perfectly symmetric around 0

(2) total area = 1

Ex: Find $P(-1.73 < Z < 0.68) = .7099$

$= .7517 - .0418$ //

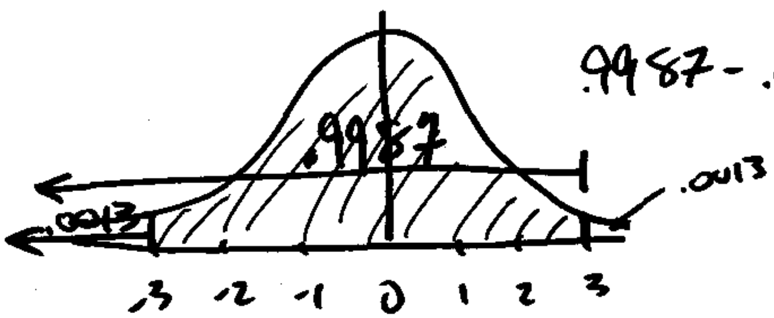


$1 - .9582 = .0418$

(6)

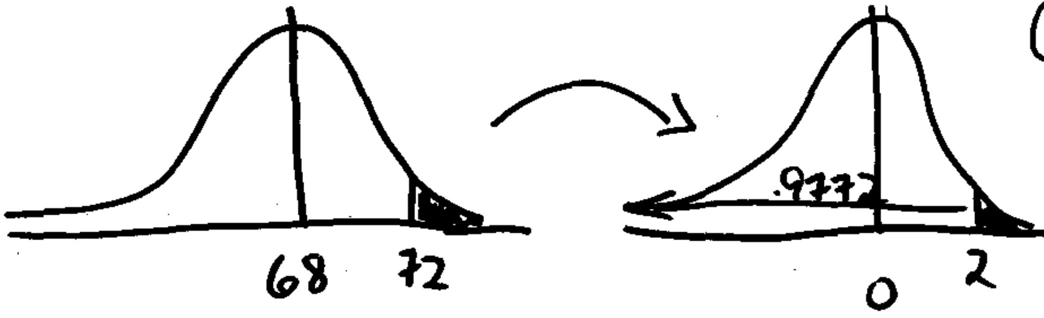
Ex: Find $P(-3 < Z < 3) = .9973$

$.9987 - .0013 = .9974$



Ex: Assume that a certain population of height is normally distributed with $\mu = 68''$ and $\sigma = 2''$.

What % of the population is $> 72''$?



X $\qquad\qquad\qquad$ Z

$P(X > 72)$ $\qquad\qquad$ $P(Z > 2)$

$$Z = \frac{X-\mu}{\sigma} \qquad \frac{72-68}{2}$$

$$P(X>72) = P(Z>2) = 1 - .9772 = \underline{.0228}$$

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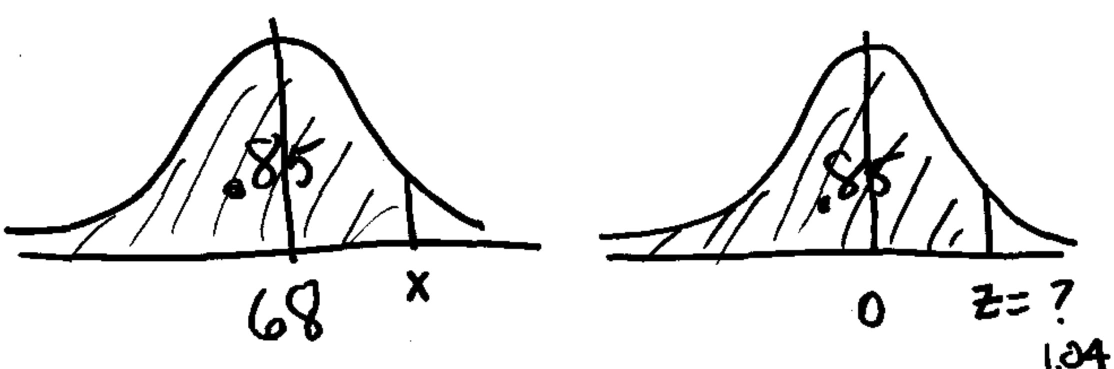
Ex.: Assume that speeds on I5 are normally distributed with $\mu = 68$ mph + $\sigma = 4$ mph

Find the 85^{th} percentile of the speeds.

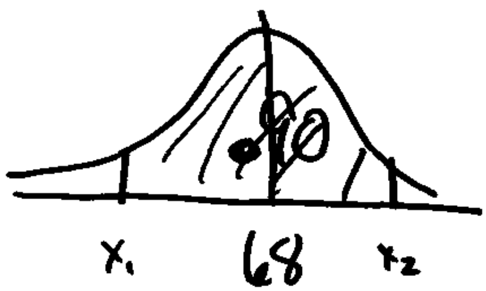


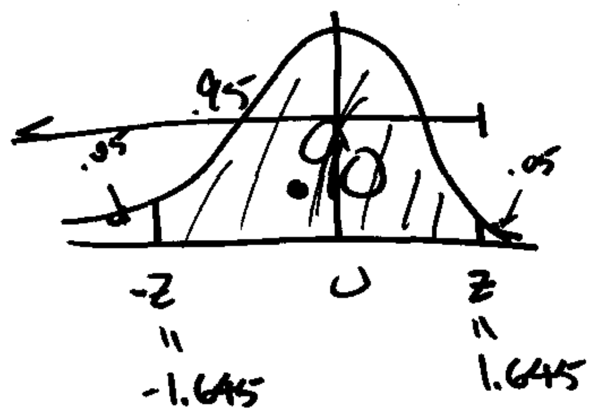
$$\frac{X-\mu}{\sigma} = z \iff x = \mu + \sigma z$$

(9)

So $x = 68 + 4(1.04) = 72.16$

Same example, but find the 2 speeds that trap the middle 90%.





$x_1 = 68 - 1.645(4) = 61.42$
$x_2 = 68 + 1.645(4) = 74.58$

(10)

Verify that the area under the bell curve is 1.

$$A = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$ so $dz = \frac{1}{\sigma} dx$

$$A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

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$$A^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}\, dw$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(z^2+w^2)}\, dz\, dw$$

Note: $\frac{A}{2} = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$

and $\frac{A^2}{4} = \int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(z^2+w^2)}\, dz\, dw$

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let $z = r\cos\theta$
$w = r\sin\theta$

Then $z^2 + w^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r$$

So $\frac{A^2}{4} = \int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2}\, r\, dr\, d\theta$

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Let $u = -\frac{1}{2}r^2$

$du = -r\,dr$

$$\frac{A^2}{4} = \int_0^{\frac{\pi}{2}} \frac{1}{2\pi} \int_0^{-\infty} e^u (-du) \; d\theta$$

$$= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} e^u \Big|_{-\infty}^{0} \; d\theta$$

$$= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2\pi} \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2\pi} \cdot \frac{\pi}{2} = \frac{1}{4}$$

$\therefore A = 1$