Stat 451
1-30-18

Geometric Distribution

(1)

Recall the geometric series:

$$a + ar + ar^2 + ar^3 + \dots = T$$

(if $r<1$)

$$T - a = ar + ar^2 + ar^3 + \dots$$
$$= r(a + ar + ar^2 + \dots) = rT$$

$$T(1-r) = a$$
$$T = \frac{a}{1-r}$$

(2)

Run a sequence of Bernoulli trials

X = trial on which the 1^{st} success occurs

$$\boxed{\begin{array}{l} X = 1, 2, \dots \\ p(x) = q^{x-1}p \end{array}}$$

Check: $\displaystyle\sum_{x=1}^{\infty} q^{x-1}p$

$$= p(1 + q + q^2 + q^3 + \dots)$$

$$= p \cdot \frac{1}{1-q} = \frac{p}{p} = 1 \checkmark$$

(3)

Find $\mu = E[X] = \sum_{\text{all } x} x\,p(x) = \sum_{x=1}^{\infty} x\,q^{x-1}p$

Let $y = x-1$ $\quad = \sum_{y=0}^{\infty} (y+1) q^y p$

$$= \underbrace{\sum_{y=0}^{\infty} y q^y p}_{1} + \underbrace{\sum_{y=0}^{\infty} q^y p}_{2}$$

1 : $q \sum_{y=1}^{\infty} y q^{y-1} p = q\mu$

2 $p(1+q+q^2+\dots) = \frac{p}{1-q} = 1$

(4)

So $\mu = q\mu + 1$

$\mu(1-q) = 1 \qquad \therefore \mu = \frac{1}{1-q} = \frac{1}{p}$

To find σ^2, compute $E[X^2] - \mu^2$

$$= \sum_{\text{all } x} x^2 p(x) - \frac{1}{p^2}$$

$$= \sum_{x=1}^{\infty} x^2 q^{x-1} p - \frac{1}{p^2}$$

$$= \dots = \frac{q}{p^2}$$

Negative Binomial (Pascal) Distribution

Same setup as on the geometric distribution, but now, $X =$ trial on which r^{th} success occurs

$$X = r, r+1, r+2, \ldots$$

$$\begin{aligned} p(x) &= P(r^{th} \text{ success occurs on trial } \# x) \\ &= P(r-1 \text{ successes in } 1^{st}\ x-1 \text{ trials} \cap \text{success on trial } x) \\ &= \binom{x-1}{r-1} p^{r-1} q^{x-r} \cdot p \end{aligned}$$

$$\boxed{p(x) = \binom{x-1}{r-1} p^{r} q^{x-r}, \quad x = r, r+1, \ldots} \quad (6)$$

$X =$ trial on which r^{th} success occurs

$$= X_1 + X_2 + \ldots + X_r$$

X_1: trial on which 1^{st} success occurs

X_2: trial on which 2^{nd} success, with the counter reset after X_1.

$$\begin{aligned} \mu &= E[X] \\ &= E[X_1] + \ldots + E[X_r] \\ &= \frac{1}{p} + \ldots + \frac{1}{p} \\ &= \frac{r}{p} \end{aligned}$$

(7)

$$\sigma^2 = V[X] = V[X_1 + \dots + X_r]$$
$$= V[X_1] + \dots + V[X_r] \quad \text{(by independence)}$$
$$= \frac{q}{p^2} + \dots + \frac{q}{p^2} = \frac{rq}{p^2}$$

Example: Each attempt has a 20% success chance. It takes 3 success to eliminate our target. Assume independence.

a) Find the prob. mass. fn.

b) Find μ

c) Find $P(X=10)$

(8)

a) $NB(r=3, p=.2)$

$$p(x) = \binom{x-1}{2}(.2)^3(.8)^{x-3} \qquad x = 3, 4, \dots$$

b) $\mu = \frac{r}{p} = \frac{3}{.2} = 15$

c) $p(10) = \binom{9}{2}(.2)^3(.8)^7 = .0604$

Add another part: Find the prob. that it takes 20 or fewer trials to get 3 successes

$$P(X \le 20) = p(3) + p(4) + \dots + p(20)$$

Another calculus review:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad \text{Let } y=\left(1+\frac{x}{n}\right)^n$$

$$\text{Then } \ln y = n\ln\left(1+\frac{x}{n}\right)$$

$$= \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{\left(1+\frac{x}{n}\right)}\cdot\frac{-x}{n^2}}{-\frac{1}{n^2}} \quad \text{by L'Hôpital}$$

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$$= \lim_{n\to\infty}\frac{x}{1+\frac{x}{n}} = x$$

$$\lim_{n\to\infty}\ln y = x$$

$$\ln\left(\lim_{n\to\infty} y\right) = x$$

$$\therefore \lim_{n\to\infty} y = e^x$$

(11)

Start with the binomial distribution

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

Let $n \to \infty$ and $p \to 0$, with np held constant

Set $np = \lambda$, so $p = \frac{\lambda}{n}$

$$p(x) = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

(12)

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \frac{\lambda^x}{n^x} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{n\cdot n \cdots n} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}$$

(13)

The Poisson Distribution

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, \ldots$$

Check: $$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \underbrace{\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)}_{e^{\lambda}}$$

$$= 1 \checkmark$$

$\mu = E[X] = \lambda$ ($\lambda = np$ was held constant)

$\sigma^2 = V[X] = \lambda$ ($npq = \lambda(1 - \frac{\lambda}{n})$)

(14)

Poisson process

- observe a continuous process over a fixed amount of time or space
- X counts the number of occurrences of a certain type
- observations in non-overlapping intervals are independent
- The probability of an occurrence in a given interval is proportional to the length of the interval

- In a very small interval, the probability of more than 1 occurrence is ≈ 0

Example: A fire station gets an average of 2.83 calls per day.
On a given day, find the prob. of getting 4 or fewer calls.

$$X \sim Pois(\lambda = 2.83) \qquad p(x) = \frac{2.83^x}{x!} e^{-2.83}$$

$$P(X \leq 4) = p(0) + p(1) + p(2) + p(3) + p(4) = .8429$$