Random variables

Stat 451
1-23-18

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Defn: A random variable is a variable whose value is determined by the outcome of an experiment.

A random variable is a mapping (function) from the sample space S into the real numbers.

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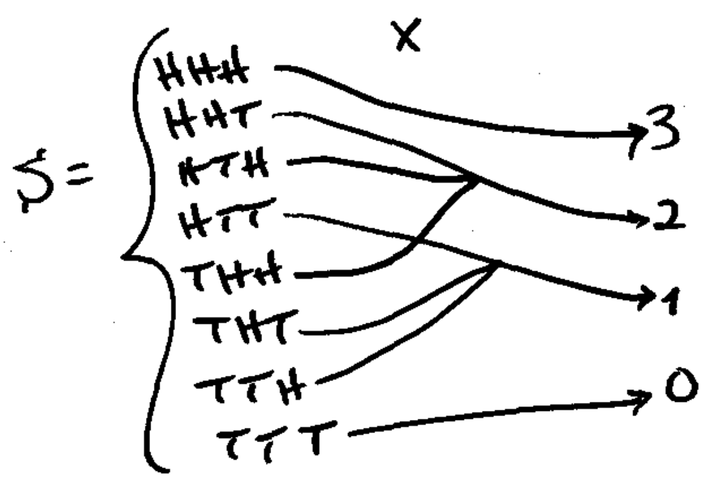
Example: Flip 3 coins.

Let X = # heads



The domain is S and the range is a subset of $\mathbb{R}$

③

If the range of X is finite or countably infinite, then X is <u>discrete</u>

If the range of X is an interval, then X is <u>continuous</u>

<u>Discrete random variables</u>

Define a <u>probability mass function</u>

$$p(x) = P(X = x)$$

④

Find $p(x)$ for our 3 coin example.

x	$p(x)$
0	1/8
1	3/8
2	3/8
3	1/8
	1

Note: $0 \le p(x) \le 1$

$$\sum_{\text{all } x} p(x) = 1$$

Example:

Roll 2 dice. Let X = sum of the dice

$$S = \left\{ \begin{matrix} 11 & 12 & \cdots & 16 \\ 21 & 22 & \cdots & 26 \\ \vdots & & & \\ 61 & 62 & \cdots & 66 \end{matrix} \right\}$$

x	$p(x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36
	1

⑤

<u>Example</u>: Roll 2 dice, but let Y = smaller of the two

y	$p(y)$
1	11/36
2	9/36
3	7/36
4	5/36
5	3/36
6	1/36
	1

$$p(y) = \frac{2(6-y)+1}{36}$$

$y = 1, \ldots, 6$

⑥

<u>Defn</u>. For a discrete random variable X, the <u>expected value</u> of X is

$$\mu = E(X) = \sum_{\text{all } x} x\,p(x) \qquad \begin{array}{c}(1^{\text{st}}\text{ moment})\\ (\text{mean})\end{array}$$

For the 3-coin example,

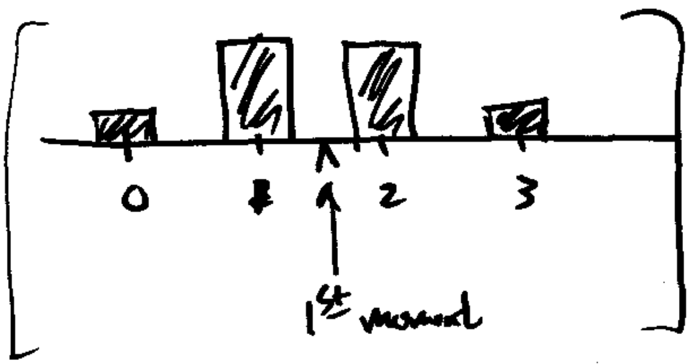
$$\mu = \sum x\,p(x) = 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8}$$

$$= \frac{0+3+6+3}{8} = \frac{12}{8}$$

$$= 1.5$$

Defn: $E(X^2)$ is the 2nd moment (7)

$$E(X^2) = \sum_{\text{all } x} x^2 p(x)$$



In our 3 coin example,

$$E[X^2] = \sum x^2 p(x)$$

$$= 0^2 \cdot \tfrac{1}{8} + 1^2 \cdot \tfrac{3}{8} + 2^2 \cdot \tfrac{3}{8} + 3^2 \cdot \tfrac{1}{8}$$

$$= \frac{0+3+12+9}{8}$$

$$= \frac{24}{8} = 3$$

(8)

Defn: $$E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

= the variance of X

It can be written σ^2 or $\text{Var}(X)$
or $V(x)$

Note: $\sigma^2 = V(x) = \sum (x-\mu)^2 p(x)$

$$= \sum (x^2 - 2\mu x + \mu^2) p(x)$$

$$= \sum x^2 p(x) - 2 \sum \mu x \, p(x) + \sum \mu^2 p(x)$$

$$= E(X^2) - 2\mu \underbrace{\sum x p(x)}_{\mu} + \mu^2 \underbrace{\sum p(x)}_{1}$$ 9

$$\sigma^2 = E(X^2) - \mu^2$$

For the 3 coins,

$$\sigma^2 = 3 - (1.5)^2 = 3 - 2.25 = .75$$

Defn: The standard deviation of X is $\sqrt{\sigma^2}$

For the 3 coins, $\sigma = \sqrt{.75} = .866$

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In the dice example where Y = smaller value, find μ.

$$E(Y) = \frac{1 \cdot 11 + 2 \cdot 9 + 3 \cdot 7 + 4 \cdot 5 + 5 \cdot 3 + 6 \cdot 1}{36}$$

$$= \frac{11 + 18 + 21 + 20 + 15 + 6}{36}$$

$$= \frac{91}{36}$$

Some properties of μ: $E(aX+b) = \sum_{\text{all } x} (ax+b)\, p(x)$

(11)

$$= \sum a x\, p(x) + \sum b\, p(x)$$

$$= a \underbrace{\sum x\, p(x)}_{E(X)} + b \underbrace{\sum p(x)}_{1}$$

$$\therefore\ E(aX+b) = a\,E(X) + b$$

(The expectation is a linear operator)

Properties of σ^2:

$$V[aX+b] = E[(aX+b)^2] - [E(aX+b)]^2$$

$$= E[a^2X^2 + 2abX + b^2] - [a\,E(X) + b]^2$$

(12)

$$= a^2E[X^2] + 2ab\,E[X] + b^2$$

$$- [a^2(E(x))^2 + 2ab\,E(X) + b^2]$$

$$= a^2\left(E(X^2) - (E(X))^2\right)$$

$$V[aX+b] = a^2\,V(X)$$

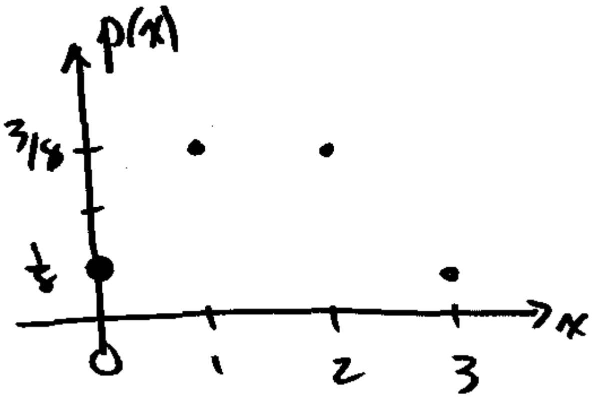
(The variance is a quadratic operator)

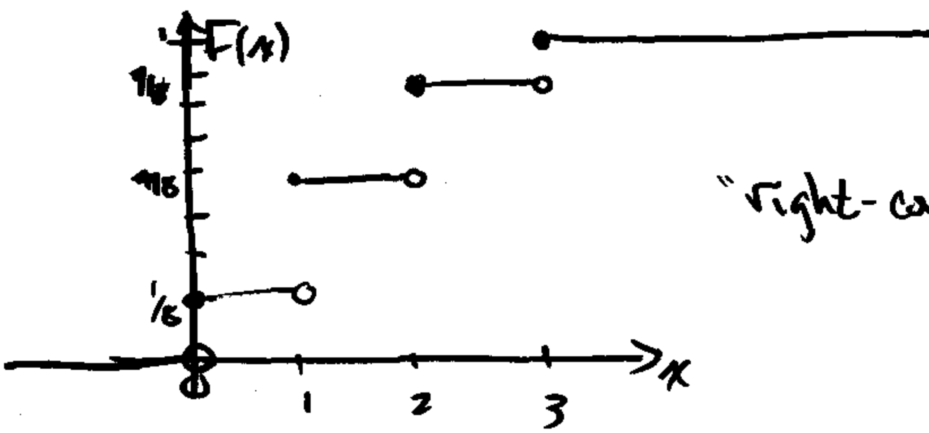
Defn: The <u>cumulative</u> <u>distribution</u> <u>function</u> (cdf) for a random variable X is

$$F(x) = P(X \le x)$$

For the 3 coins:

x	$p(x)$ (pmf)	$F(x)$ (cdf)
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	8/8





"right-continuous"