Continuation of counting rules

4 Combination rule

Start with n items. Select r of them, without keeping track of order

Example: A, B, C, D, E Pick 3

ABC (ACB, BAC, BCA, CAB, CBA)
ABD
ABE
ACD
ACE
ADE
BCD
BCE
BDE
CDE

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$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!}$$

"n choose r"

Example: Deal 5 cards from a deck of 52
How many combinations are possible?

$$C_{52,5} \text{ or } \binom{52}{5} = \frac{52!}{5!\,47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

5 Arrangements of like objects

Suppose there are n items of k different types, with n_1 of type 1, n_2 of type 2, etc.

$$(n_1 + n_2 + \ldots + n_k = n).$$

The # of different possible arrangements is

$$\binom{n}{n_1 \; n_2 \cdots n_k} = \frac{n!}{n_1! \, n_2! \cdots n_k!}$$

Example: STATISTICS

How many different ways can the letters be arranged?

$n = 10$

Type 1: A $\quad n_1 = 1$

Type 2: C $\quad n_2 = 1$

Type 3: I $\quad n_3 = 2$

Type 4: S $\quad n_4 = 3$

Type 5: T $\quad n_5 = 3$

$\overline{n = 10}$

$$\binom{10}{1 \; 1 \; 2 \; 3 \; 3} = \frac{10!}{1!\,1!\,2!\,3!\,3!} = \frac{10!}{72} = 50{,}400$$

<u>Example</u>: Flip a coin, then roll a 6-sided die, then select 2 cards from a deck. (order doesn't matter)
How many outcomes are possible?

$$N = 2 \times 6 \times \binom{52}{2}$$

$$= 12 \cdot \frac{52!}{2!\,50!} = 12\ \frac{52 \cdot 51}{2} = 15{,}912$$

<u>Example</u>: How many 10-digit phone numbers are possible, if the 1^{st} digit cannot be a 0 or 1, + the 2^{nd} digit must be 0 or 1

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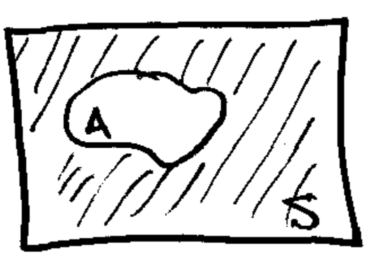
$$\underline{8} \times \underline{2} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10}$$

$$= 16 \times 10^8$$

<u>Defn</u>: The <u>complement</u> of an event A is the set A^c consisting of all items in S that are <u>not</u> in A.

A^c:



$$N(A) + N(A^c) = N$$

$$\frac{N(A)}{N} + \frac{N(A^c)}{N} = 1$$

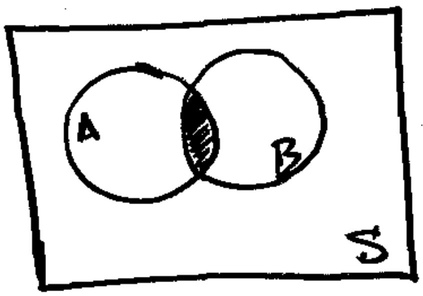
$$P(A) + P(A^c) = 1$$

(7)

Complement rule: $\boxed{P(A^c) = 1 - P(A)}$

Defn: The intersection of events A and B is $A \cap B$, which is the set of items appearing in both A and B.

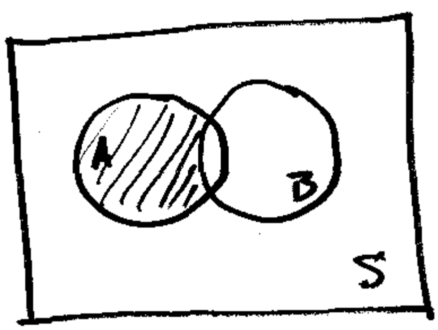
$A \cap B$:



(8)

Defn: $A - B = A \cap B^c$

$A - B$:

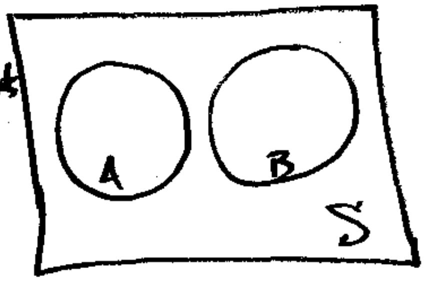


Defn: A and B are disjoint or mutually exclusive if $A \cap B = \emptyset$

↑ null set or empty set

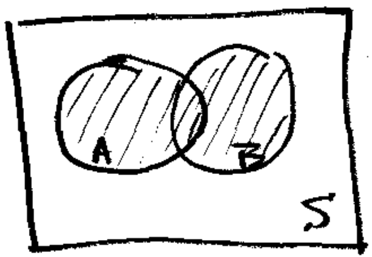
(9)

Disjoint events A and B



Defn: The union of events A and B is the set of items $A \cup B$ that appear in either A or B (or possibly both)

$A \cup B$:



(10)

Suppose A & B are disjoint.

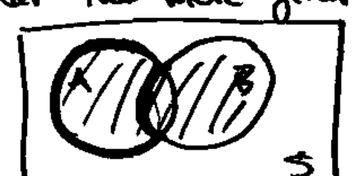
Then $N(A \cup B) = N(A) + N(B)$

$$\frac{N(A \cup B)}{N} = \frac{N(A)}{N} + \frac{N(B)}{N}$$

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Union rule for disjoint events

Consider the more general case



$A \cup B = (A - B) \cup B$

disjoint

(11)

$$P(A \cup B) = P(A - B) + P(B)$$

Also, $A = (A - B) \cup (A \cap B)$

disjoint

$$P(A) = P(A - B) + P(A \cap B)$$

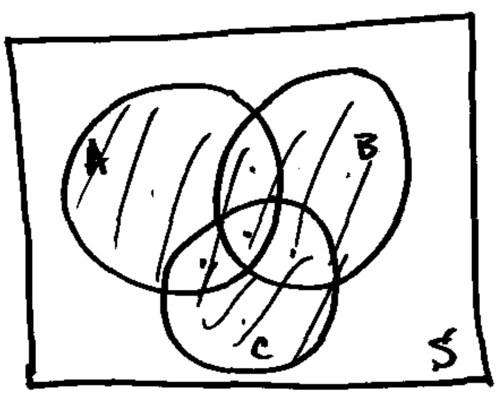
$$P(A - B) = P(A) - P(A \cap B)$$

$$\therefore \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

<u>Union rule</u>

(12)

Consider 3 events:



$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

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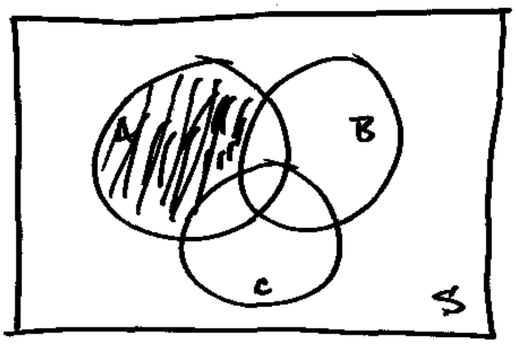
Example: $A - (B \cup C)$

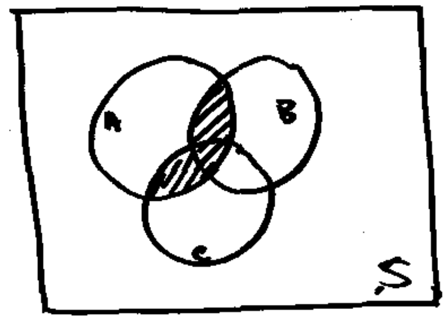
"in A and not in B or C"



Example: $(A \cap B) \cup (A \cap C)$

"in A and B or in A and C"

$$A \cap (B \cup C)$$



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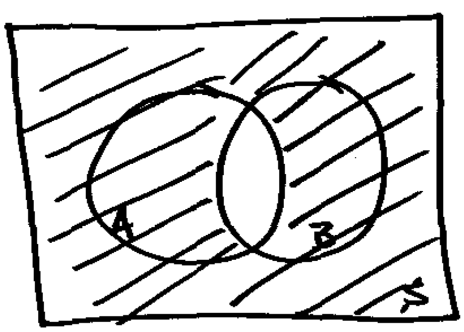
Example: $(A \cap B)^c$

"not in both A & B"



"not in A or not in B" $A^c \cup B^c$

De Morgan's Law