Stat 451
11-15-16

Conf. int. for μ: $\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$

(1)

— If n is large OR if the original distribution is approximately normal

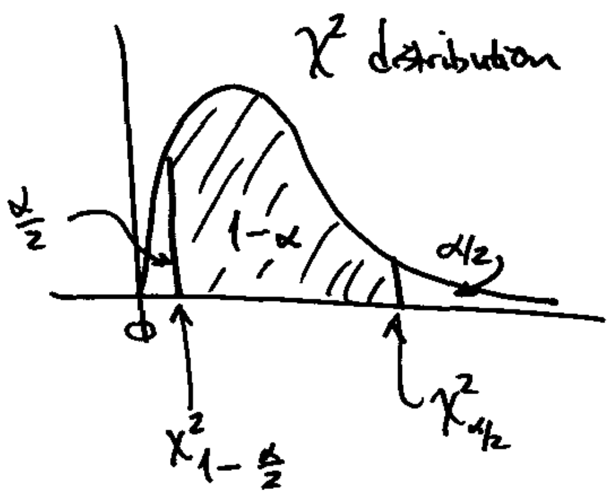
Conf. int. for p: $\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n}}$

— If $n \geq 5$

Conf. int. for σ^2 or σ

Fact: $\frac{(n-1)s^2}{\sigma^2}$ has a chi-squared (χ^2) distribution with $n-1$ df, provided that the sample came from a normal distribution.

(2)



$$P\left(\chi^2_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2} \right) = 1-\alpha$$

$$\chi^2_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2}$$

③

$$\frac{1}{\chi^2_{1-\alpha/2}} > \frac{\sigma^2}{(n-1)s^2} > \frac{1}{\chi^2_{\alpha/2}}$$

$$\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} > \sigma^2 > \frac{(n-1)s^2}{\chi^2_{\alpha/2}}$$

$$\boxed{\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}$$

$$\boxed{\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}}$$

④

Example: In a sample of 20 measurements of items produced on an assembly line, the sample standard deviation is 2mm. Find a 95% C.I. for the population std. dev. Assume that the measurements are normally distributed

$df = 19$ $\quad 1-\alpha = .95$ so $\alpha = .05$,

$\alpha/2 = .025$, $1-\frac{\alpha}{2} = .975$

$\chi^2_{.025} = 32.852$ $\quad \chi^2_{.975} = 8.907$

⑤

C.I. for σ: $\left(\sqrt{\dfrac{(n-1)s^2}{\chi^2_{.025}}}, \sqrt{\dfrac{(n-1)s^2}{\chi^2_{.975}}} \right)$

$= \left(\sqrt{\dfrac{19(4)}{32.852}}, \sqrt{\dfrac{19(4)}{8.907}} \right)$

$= (1.52, 2.92)$

Estimate is $s = 2$

⑥

What if the parameter that you need to estimate is not μ, σ^2, or p?

Example: $f(x) = \lambda e^{-\lambda x}$

Method 1: Method of moments (MOM)

Equate the first sample moment with the first population moment + solve for the parameter of interest.

That is, set $\bar{x} = \mu$ & solve for the unknown parameter

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For the exponential distribution,

$$\mu = \frac{1}{\lambda}$$

Set $\frac{1}{\lambda} = \bar{x}$

& solve for λ:

$$\hat{\lambda} = \frac{1}{\bar{x}}$$

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x) = \int_{0}^{\infty} x \lambda e^{-\lambda x}\, dx = \dots = \frac{1}{\lambda}$$

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Example. $p(x) = \dfrac{\alpha^x e^{-\alpha}}{x!}$ (Poisson)

$$\mu = \alpha$$

MOM: Set $\alpha = \bar{x}$

$$\hat{\alpha} = \bar{x}$$

Example. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

There are 2 parameters: μ, σ^2

Also must equate the 2^{nd} sample moment to the 2^{nd} population moment

Sample moments   population moments   (9)

1^{st}	$\overline{X}$	μ
2^{nd}	$\frac{1}{n}\sum x_i^2$	$E[X^2]$

Set $\bar{x} = \mu$ $\quad \therefore \hat{\mu} = \bar{x}$

Set $\frac{1}{n}\sum x_i^2 = E[X^2]$

$\sigma^2 = E[X^2] - \mu^2$

$E[X^2] = \sigma^2 + \mu^2$

$$\frac{1}{n}\sum x_i^2 = \sigma^2 + \mu^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum x_i^2 - \mu^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\left(\sum x_i^2 - n\bar{x}^2\right) = \frac{(n-1)s^2}{n}$$ (10)

Method 2 : Maximum Likelihood Estimation (MLE)

Start with the joint probability or density function of $X_1, \ldots, X_n$, assuming independence.

$$f(x_1, \ldots, x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n)$$
$$= \prod_{i=1}^{n} f(x_i)$$

Treat this as a function of your unknown parameter.

(11)

Call it the "likelihood" function

$$\text{lik}(\theta) = \prod_{i=1}^{n} f(x_i)$$

Find the value of θ that maximizes this.

$$\ln(\text{lik}(\theta)) = \sum_{i=1}^{n} \ln f(x_i)$$

$$\parallel$$

$$\mathcal{L}(\theta)$$

"log likelihood"

Process: take the derivative of $\mathcal{L}(\theta)$ + set it equal to 0.

(12)

Ex1: $f(x) = \lambda e^{-\lambda x}$

$$\text{lik}(\underset{\lambda}{\theta}) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda^n e^{-\lambda \Sigma x_i}$$

$$\mathcal{L}(\lambda) = n \ln \lambda - \lambda \Sigma x_i$$

$$\mathcal{L}'(\lambda) = \frac{n}{\lambda} - \Sigma x_i \overset{\text{set}}{=} 0$$

$$\frac{n}{\lambda} = \Sigma x_i \qquad \hat{\lambda} = \frac{n}{\Sigma x_i} = \frac{1}{\bar{X}}$$

(13)

Ex2: $p(x) = \dfrac{\alpha^x e^{-\alpha}}{x!}$

$$\text{lik}(\alpha) = \prod_{i=1}^{n} p(x_i) = \prod_{i=1}^{n} \frac{\alpha^{x_i} e^{-\alpha}}{x_i!}$$

$$= \frac{\alpha^{\sum x_i} e^{-n\alpha}}{\prod_{i=1}^{n} x_i!}$$

$$\mathcal{L}(\alpha) = \sum x_i \ln\alpha - n\alpha - \ln \prod_{i=1}^{n} x_i!$$

$$\mathcal{L}'(\alpha) = \frac{\sum x_i}{\alpha} - n \stackrel{\text{set}}{=} 0 \qquad \hat{\alpha} = \frac{\sum x_i}{n} = \bar{x}$$

(14)

Ex3: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\text{lik}(\mu,\sigma) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma^n (2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\mathcal{L}(\mu,\sigma) = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

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$$\frac{\partial \mathcal{L}}{\partial \mu} = -\frac{1}{\cancel{2}} \sum_{i=1}^{n} \cancel{2}\left(\frac{x_i-\mu}{\cancel{\sigma}}\right)\left(-\frac{1}{\cancel{\sigma}}\right) \stackrel{\text{set}}{=} 0$$

$$\sum_{i=1}^{n} x_i - \mu \cdot n = 0$$

$$\hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial \mathcal{L}}{\partial \sigma} = -\frac{n}{\sigma} - \frac{1}{\cancel{2}} \sum_{i=1}^{n} \cancel{2}\left(\frac{x_i-\mu}{\cancel{\sigma}}\right)(x_i-\mu)\left(-\frac{1}{\sigma^2}\right) \stackrel{\text{set}}{=} 0$$

$$-n\sigma^2 + \sum_{i=1}^{n} (x_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{(n-1)s^2}{n}$$

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HW #6 p.228 #5, 10a

p.269 # 2, 3, 10, 19

due Tues 11/22