Tuesday's example, continued.

Stat 451

10-27-16

We had $f(x,y) = 8xy \quad 0 \le y \le x \le 1$

$g(x) = 4x^3, \ 0 \le x \le 1$

$h(y) = 4y - 4y^3, \ 0 \le y \le 1$

(1)

Find ρ. We need $E[XY]$, $E[X]$, $E[X^2]$, $E[Y]$, $E[Y^2]$

$$E[XY] = \int_0^1 \int_0^x xy \, 8xy \, dy\,dx = \int_0^1 \left[8x^2 \int_0^x y^2 dy\right] dx$$

$$= \int_0^1 8x^2 \left.\frac{y^3}{3}\right|_{y=0}^{x} dx = \int_0^1 8\frac{x^5}{3}\,dx$$

(2)

$$\left.8\frac{x^6}{18}\right|_0^1 = \frac{4}{9}$$

$$E[X] = \int_0^1 x\,4x^3\,dx = \left.\frac{4x^5}{5}\right|_0^1 = \frac{4}{5}$$

$$E[X^2] = \int_0^1 x^2\,4x^3\,dx = \left.\frac{4x^6}{6}\right|_0^1 = \frac{2}{3}$$

$$E[Y] = \int_0^1 y(4y - 4y^3)\,dy = 4\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1$$

$$= 4\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15}$$

(3)

$$E[Y^2] = \int_0^1 y^2(4y - 4y^3)\,dy = 4\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1$$

$$= 4\left[\frac{1}{4} - \frac{1}{6}\right] = \frac{1}{3}$$

$$\sigma_{XY} = E[XY] - E[X]\,E[Y]$$

$$= \frac{4}{9} - \frac{4}{5}\cdot\frac{8}{15} = \frac{4}{225}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$= \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$$

(4)

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2$$

$$= \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{4/225}{\sqrt{\frac{2}{75}}\sqrt{\frac{11}{225}}} = .492$$

Suppose that X & Y are independent.

Then $f(x,y) = g(x)h(y)$

If so, then $E[XY] = \iint x\,y\,f(x,y)\,dy\,dx$

$$= \int \underline{\int xy\, g(x) h(y)\, dy}\, dx$$ (5)

$$= \int \Big[x\, g(x) \underbrace{\int y\, h(y)\, dy}_{\mu_Y} \Big] dx$$

$$= \mu_Y \underbrace{\int x\, g(x)\, dx}_{\mu_X}$$

If X & Y are indep, then $E[XY] = E[X]E[Y]$

$$\therefore \sigma_{XY} = 0$$

$$\therefore \rho_{XY} = 0$$

(6)

Note: $\rho_{XY} = 0 \not\Rightarrow$ independence

$$V[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - \Big[(E[X])^2 + 2E[X]E[Y] + (E[Y])^2 \Big]$$

$$= \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}$$

Note: If X & Y are indep, then $V[X+Y] = \sigma_X^2 + \sigma_Y^2$

(7)

Let $X_1, X_2, \ldots, X_n$ be <u>independent</u> random variable, each one having mean μ and variance σ^2

Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\underbrace{E[X_i]}_{\mu} = \mu$$

$$V[\bar{X}] = V\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\left[\sum_{i=1}^{n}\underbrace{V[X_i]}_{\sigma^2} + 2\text{ times } \overset{\text{all } 0}{\cancel{\text{each cov}}}\right]$$

$$= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n} \quad \therefore \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

(8)

Midterm exam: Thursday Nov 3

Bring 1 page of notes, calculator

- Descriptive statistics
- Probability: joint, marginal, conditional
- Bernoulli, Binomial, Hypergeometric, Geometric, Neg Bino, Poisson, Uniform, Exponential, Normal
- Expectations, Variances, Covariance, Correlation, joint & marginal & conditional distributions
- Independence

You have just switched to a new cell phone plan that lets you make unlimited calls to 5 of your friends. But you have 13 friends that you frequently call. How many different ways can you choose 5 of them?

In a certain game that uses a pair of dice, a winning roll consists of getting a sum of either 7 or 11. What is the probability of a winning roll?

How many ways are there of arranging the letters in the word MATHEMATICS?

There are 25 runners competing in a race that awards distinct prizes for first, second, and third place. How many different outcomes are possible?

On the average, your web site is viewed once every 15 seconds. Find the probability that, in a 30-second interval, there are no visitors to the site.

Suppose that a company wins 30% of the contracts on which it bids. Assume independence between bids. In one month, it bids on 20 contracts. Find the probability that it wins exactly 6 of them.

The following is the joint density function of the random variables X and Y:

$$f(x,y) = \begin{cases} c(x+y), & 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}.$$

a) Find c.
b) Find $g(x)$ and $h(y)$.
c) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$.
d) Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
e) Find $\mathrm{E}(XY)$ and $\mathrm{Cov}(X,Y)$.
f) Find ρ_{XY}.
g) Find $\mathrm{E}(X+Y)$
h) Find $\mathrm{Var}(X+Y)$

Each day, 40% of the email sent to my address is spam. If an email is actually spam, the spam filter will correctly identify it 80% of the time. If the email is not spam, the spam filter still tags it as spam 5% of the time.

Complete the table with the correct joint and marginal probabilities:

		ID'ed as:		
		Spam	Not spam	Total
Actually:	Spam	0.32		0.40
	Not spam	0.03		
	Total			1.00

Given that a message has been tagged as spam, what is the probability that it is actually spam?

The following is the probability density function of the random variable X, which represents the success rate of a production batch of electronic components.

$$f(x) = \begin{cases} cx^2, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}.$$

a) Find c.

b) Find the probability that X is greater than 0.5