Stat 451
11-25-16

①

Joint probability distributions

Discrete case

Example: Roll 2 dice let X = minimum of 2 dice
Y = maximum of 2 dice

		Y						
		1	2	3	4	5	6	
X	1	$1/36$	$2/36$	$2/36$	$2/36$	$2/36$	$2/36$	$11/36$
	2	0	$1/36$	$2/36$	$2/36$	$2/36$	$2/36$	$9/36$
	3	0	0	$1/36$	$2/36$	$2/36$	$2/36$	$7/36$
	4	0	0	0	$\frac{1}{36}$	$2/36$	$2/36$	$5/36$
	5	0	0	0	0	$1/36$	$2/36$	$3/36$
	6	0	0	0	0	0	$\frac{1}{36}$	$1/36$
		$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

The body of the table contains the joint prob. distr.
$p(x,y) = P(X=x \cap Y=y)$

②

$$P(X=x) = \sum_{y} p(x,y)$$

and $$P(Y=y) = \sum_{x} p(x,y)$$

In our table, $p(5,6) = 2/36$

$$P(X=5) = 3/36 \qquad P(Y=6) = \frac{11}{36}$$

See if the events "$X=5$" and "$Y=6$" are independent.

$$P(Y=6 \mid X=5) = \frac{P(Y=6 \cap X=5)}{P(X=5)} = \frac{2/36}{3/36} = \frac{2}{3}$$

$$\neq P(Y=6) \qquad \therefore \text{They are not independent}$$

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In general, the conditional probability of X, given Y is $h(x|y) = \dfrac{p(x,y)}{p(y)}$

Example: A team plays 3 games, with independent outcomes

They win with prob. .5
lose .4
tie .1

X = # wins Y = # losses

Construct the joint & marginal prob. distribs.

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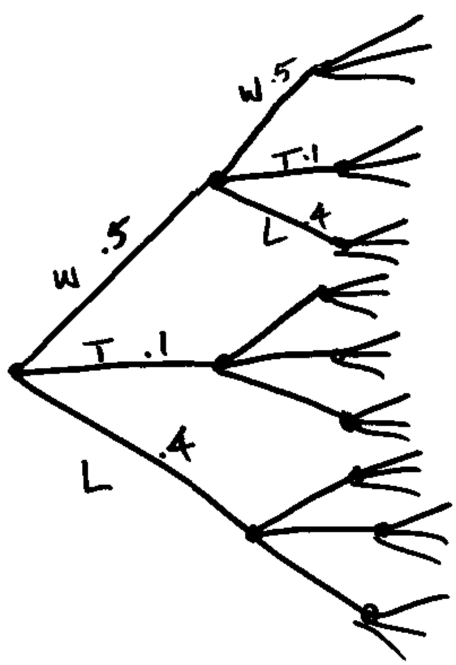

Y = losses

X wins	0	1	2	3	
0	$(.1)^3$	$3(.1)^2(.4)$	$3(.1)(.4)^2$	$(.4)^3$	.125
1	$(.5)(.1)^2 3$	$.5(.1)(.4)$ (6)	$(.5)(.4)^2$ (3)	0	.375
2	$3(.5)^2(.1)$	$3(.5)^2(.4)$	0	0	.375
3	$(.5)^3$	0	0	0	.125
	.216	.432	.288	.064	1

Find the conditional prob. of getting 3 wins, given that there are no losses. $h(3|0) = \dfrac{p(3,0)}{P(Y=0)} = \dfrac{.125}{.216} = .579$

⑤

From the pattern, we can see that

$$p(x,y) = \frac{3!}{x!\,y!\,(3-x-y)!} (.5)^x (.4)^y (.1)^{3-x-y}, \quad 0 \le x+y \le 3$$

The joint density function for 2 continuous random variables

Example: $f(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

Probabilities are volumes underneath this surface. And the total volume must be 1.

⑥

Check the total volume:

$$\underbrace{\int_0^1 \int_0^x 8xy \, dy \, dx} \quad \text{OR} \quad \int_0^1 \int_y^1 8xy \, dx \, dy$$

$$= \int_0^1 8x \left. \frac{y^2}{2} \right|_{y=0}^{x} dx$$

$$= \int_0^1 4x(x^2-0)\, dx = \int_0^1 4x^3 dx = x^4 \Big|_0^1$$

$$= 1-0$$

$$= 1 \quad \checkmark$$

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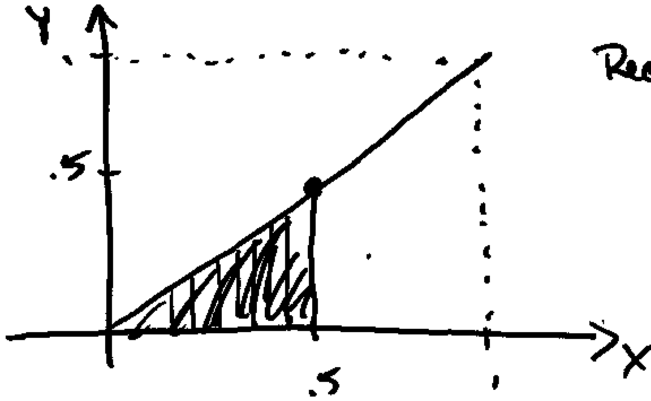
Find the prob. that both X and Y are less than .5



Recall $f(x,y) = 8xy$ for $0 \le y \le x \le 1$

$$P(X<.5 \cap Y<.5) = \int_0^{.5}\int_0^{x} 8xy \, dy \, dx$$

$$= \int_0^{.5} 8x \left.\frac{y^2}{2}\right|_{y=0}^{x} dx = \int_0^{.5} 4x(x^2-0)\,dx$$

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$$= \int_0^{.5} 4x^3\,dx = \left. x^4 \right|_0^{.5} = (.5)^4 - 0 = \frac{1}{16}$$

If $f(x,y)$ is the joint density, then the marginal density of X is

$$g(x) = \int_{\substack{\text{all } y \\ \text{(in terms of } x)}} f(x,y)\,dy$$

and the marginal density of Y is

$$h(y) = \int_{\substack{\text{all } x \\ \text{(in terms of } y)}} f(x,y)\,dx$$

(9)

$$g(x) = \int_0^x 8xy \, dy$$

$$= 8x \frac{y^2}{2}\Big|_{y=0}^{x} = 4x(x^2-0) = 4x^3 \;, \quad 0 \le x \le 1$$

$$h(y) = \int_y^1 8xy \, dx = 8y \frac{x^2}{2}\Big|_{x=y}^{1}$$

$$= 4y(1^2 - y^2) = 4y - 4y^3, \quad 0 \le y \le 1$$

Check: $\int_0^1 4y - 4y^3 \, dy = \left(\frac{4y^2}{2} - \frac{4y^4}{4}\right)\Big|_0^1 = 1$

(10)

Def$\underline{\text{n}}$: The random variables X & Y are independent if $h(x|y) = g(x) \quad \forall x, y$

$$h(x|y) = \frac{f(x,y)}{h(y)}$$

Equivalently, X & Y are indep if

$$f(x,y) = g(x)\, h(y)$$

In our example, is $8xy \stackrel{?}{=} 4x^3(4y - 4y^3)$ $\underline{\text{No}}$

$\therefore$ X & Y are dependent.

(11)

Expectations

$$E[XY] = \iint_{\text{all } (x,y)} xy\, f(x,y)\, dx\, dy$$

To find $E[X], E[X^2], E[Y], E[Y^2]$, first find the marginal density + then use our previous definition.

Defn: The covariance of X and Y is

$$Cov(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

(12)

$$= E[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$$

$$= E[XY] - \mu_Y \underbrace{E[X]}_{\mu_X} - \mu_X \underbrace{E[Y]}_{\mu_Y} + \mu_X\mu_Y$$

$$\sigma_{XY} = E[XY] - \mu_X\mu_Y$$

Defn: The correlation between X & Y is

$$Corr(X,Y) = \rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$

If you are given a joint density function $f(x,y)$, how do you find the correlation?

(1) Find both marginal densities

(2) From the marginals, find $E[X]$, $E[X^2]$, $E[Y]$, $E[Y^2]$

(3) Compute σ_X^2 & σ_Y^2

(4) From the joint density, find $E[XY]$

(5) Compute σ_{XY}

(6) Compute ρ_{XY}