Stat 451
10-20-16

The "memory-less" property of the exponential distribution:

1

$$P(X > s+t \mid X > s) = P(X > t)$$

Recall: $f(x) = \lambda e^{-\lambda x} \quad x > 0$

and $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

$$P(X > x) = 1 - F(x) = e^{-\lambda x}$$

2

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)}$$

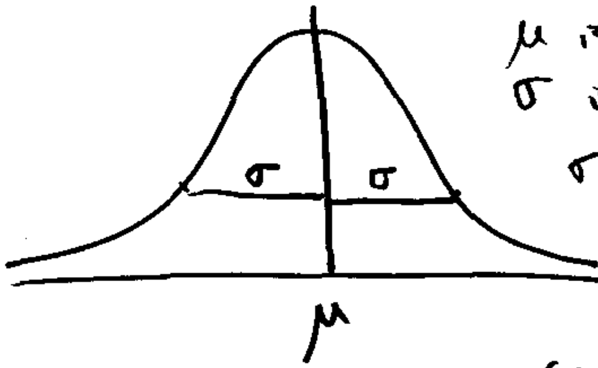
$$= \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P(X > t)$$

3

The normal distribution



μ is the mean
σ is the std. dev.
σ is also the distance from μ to the points of inflection

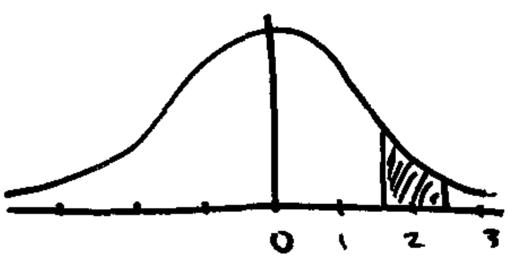
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

The standard normal distribution has $\mu = 0$ and $\sigma = 1$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

4

Example. In the standard normal dist., find the prob. of seeing a value between 1.57 and 2.5



$$P(1.57 < z < 2.5) = \int_{1.57}^{2.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$$

$$= F(z)\Big|_{1.57}^{2.5}$$

$$= F(2.5) - F(1.57)$$

$$= .9938 - .9418$$

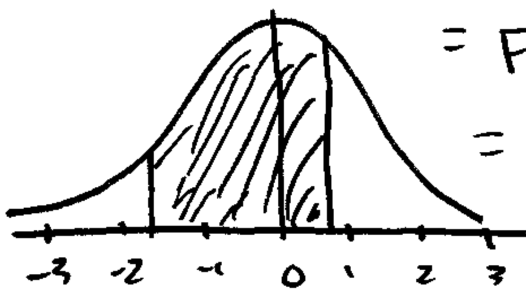
$$= .052$$

5

2 important properties of the standard normal dist: ① perfectly symmetric around 0

② total area = 1

Ex: Find $P(-1.73 < Z < 0.68)$



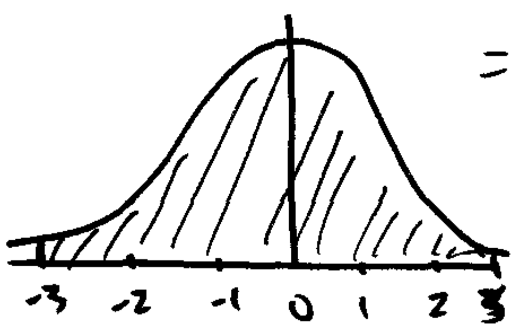
$= F(0.68) - F(-1.73)$

$= F(0.68) - [1 - F(1.73)]$

$= .7099$

6

Ex: Find $P(-3 < Z < 3)$



$= F(3) - F(-3)$

$= F(3) - [1 - F(3)]$

$= 2F(3) - 1$

$= .9973$

Find $P(-2 < Z < 2) = F(2) - F(-2)$

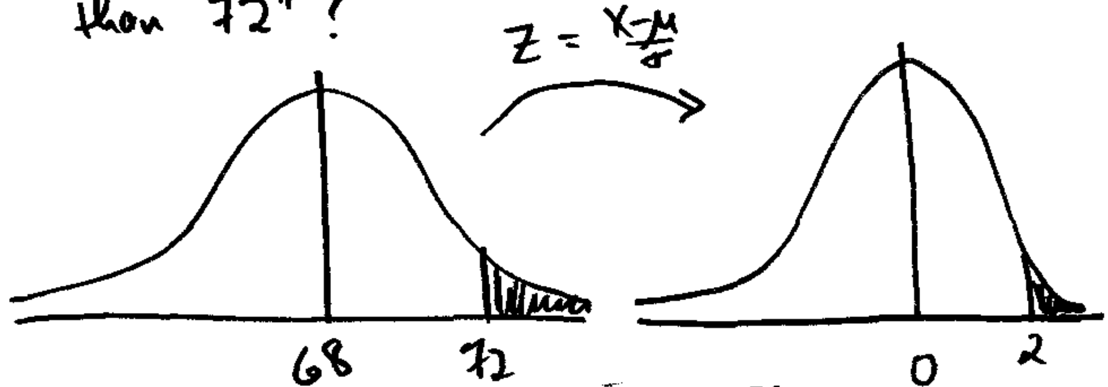
$= 2F(2) - 1$

$= .9545$

7

Example: Assume that in a certain population, heights are normally distributed with $\mu = 68''$ and $\sigma = 2''$

What percentage of the population is taller than $72''$?

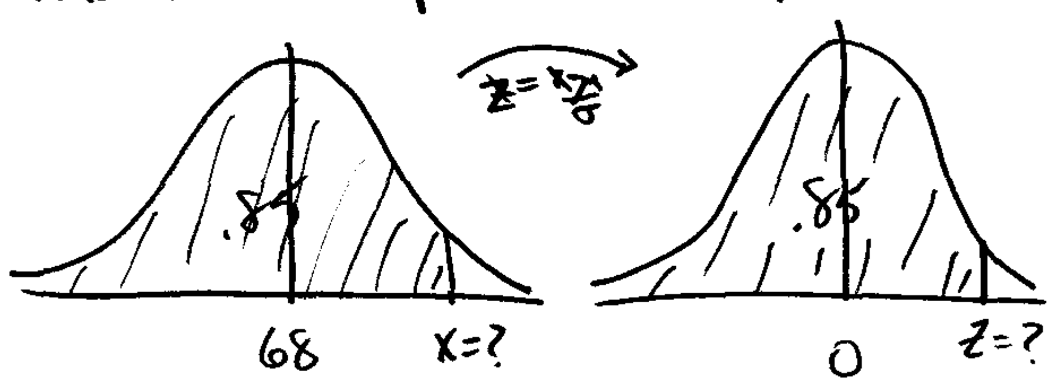


$= 1 - F(2) = .0228$

8

Ex: Assume that speeds are normally distributed with $\mu = 68$ and $\sigma = 4$

Find the 85^{th} percentile of the speeds.

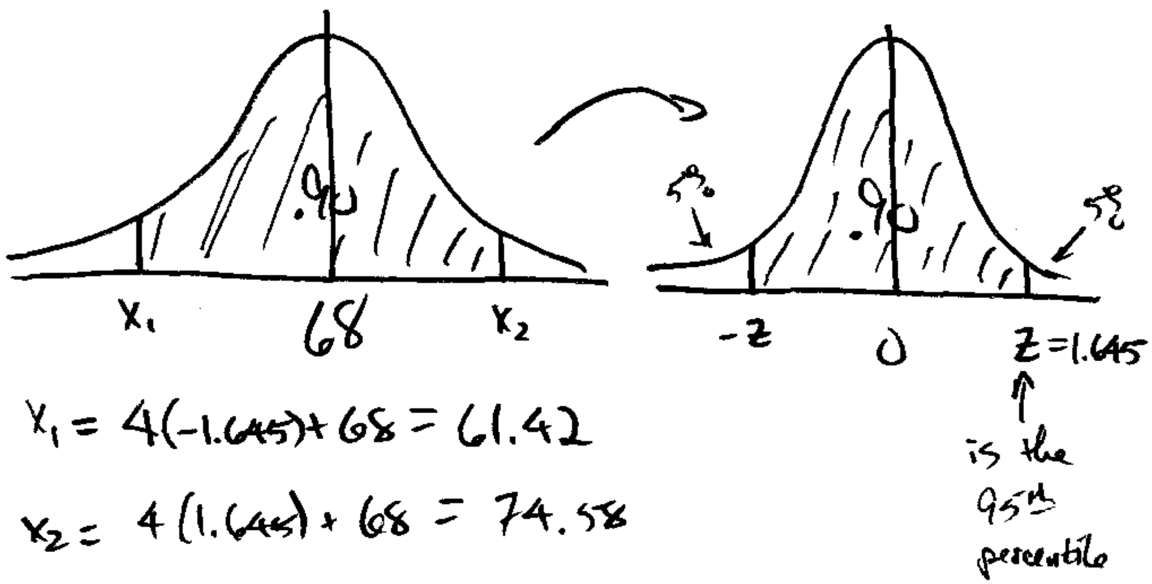


$Z = 1.036$

$x = \sigma z + \mu$

$X = 4(1.036) + 68$
$= 72.15$

Same example, but find the 2 speeds $(68 \pm ?)$ that trap the middle 90% of all speeds (9)



$x_1 = 4(-1.645) + 68 = 61.42$

$x_2 = 4(1.645) + 68 = 74.58$

(10)

Verify that the total area under the bell curve is 1.

$$A = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$

So $dz = \frac{1}{\sigma} dx$

$$A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

(11)

$$A^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} dw$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(z^2+w^2)} dz\, dw$$

Then note that $A/2 = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$

So $\frac{A^2}{4} = \int_0^{\infty}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(z^2+w^2)} dz\,dw$

(12)

Let $z = r\cos\theta$
$w = r\sin\theta$

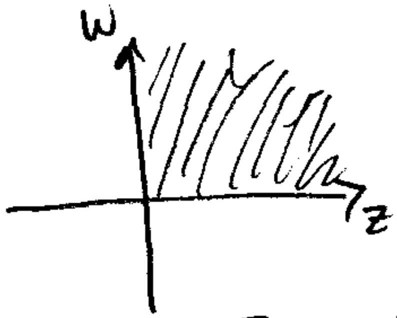
Then $z^2+w^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2$

$$J = \begin{vmatrix} \partial z/\partial r & \partial z/\partial\theta \\ \partial w/\partial r & \partial w/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$
$$= r(\cos^2\theta + \sin^2\theta) = r$$

(13)

$$\frac{A^2}{4} = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \underbrace{r}_{J}\, dr\, d\theta$$



let $u = -\frac{1}{2}r^2$

$du = -r\,dr$

$$\frac{A^2}{4} = \int_0^{\frac{\pi}{2}} \int_0^{-\infty} \frac{1}{2\pi} e^{u} (-du)\, d\theta$$

(14)

$$= \frac{1}{2\pi} \int_0^{\pi/2} \underbrace{\int_{-\infty}^{0} e^{u}\, du}_{\substack{e^{u}\big|_{-\infty}^{0} \\ 1 - 0}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2\pi}\, \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2\pi}\left(\frac{\pi}{2}\right) = \frac{1}{4}$$

$$\frac{A^2}{4} = \frac{1}{4} \quad \therefore \quad A^2 = 1 \qquad \therefore A = 1$$

HW #4 due 10/27

p.156 # 1, 2, 5, 7, 10

Reminder: midterm 1 week after that